Six combinatorial classes of maximal convex tropical tetrahedra

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AMS class.: 15A80; 52B10; 14T05.

Keywords and phrases: combinatorial class, tropical tetrahedron, convex 3–dimensional body, circulant matrix.

Short title: Six classes of maximal convex tropical tetrahedra

Abstract

In this paper we bring together tropical linear algebra and convex 3–dimensional bodies. We show how certain convex 3–dimensional bodies having 20 vertices and 12 facets can be encoded in a 4×4 integer zero–diagonal matrix A. A tropical tetrahedron is the set of points in $\mathbb{R}^3$ tropically spanned by four given tropically non–coplanar points. It is a near–miss Johnson solid. The coordinates of the points are arranged as the columns of a 4×4 real matrix A and the tetrahedron is denoted $\operatorname{span}(A)$. We study tropical tetrahedra which are convex and maximal, computing the extremals of $\operatorname{span}(A)$ and the length (tropical or Euclidean) of its edges. Then, we classify convex maximal tropical tetrahedra, combinatorially. There are six classes, up to symmetry and chirality. Only one class contains symmetric solids and only one contains chiral ones. In the way, we show that the combinatorial type of the regular dodecahedron does not occur here. We also prove that convex maximal tropical tetrahedra are not vertex–transitive, in general. We give families of examples, circulant matrices providing examples for two classes. A crucial role is played by the 2×2 minors of A.

1 Introduction

Here $\oplus = \max$, $\odot = +$ are the tropical operations: addition and multiplication. In classical mathematics, a tetrahedron is the span of four non–coplanar points in 3–dimensional space. We have a choice to make: affine or projective geometry. In tropical mathematics, a tetrahedron should be the tropical span of four tropically non–coplanar points in 3–dimensional space. It is known that this set is neither pure–dimensional nor convex, in general.

*Partially supported by UCM research group 910444. Corresponding author

Here we study tropical tetrahedra which are both convex and maximal. The coordinates of the points are arranged as the columns of a 4×4 real matrix A and the tetrahedron is denoted $\operatorname{span}(A)$. A leading role is played by the 2×2 minors of A. Firstly, because they provide the tropical length of the edges of $\operatorname{span}(A)$ (theorem 5). Secondly, because they characterize maximality of $\operatorname{span}(A)$ (lemma 6). The 2×2 minors of A also yield the types of the tropical lines $L_{ij} := L(\operatorname{col}(A, i), \operatorname{col}(A, j))$ and, with the type information we are able compute the coordinates of all the extremals of $\operatorname{span}(A)$.

In section 3, we show that the f–vector (i.e., vector whose components are the number of vertices, edges and facets) of a maximal convex $\operatorname{span}(A)$ is $(20, 30, 12)$, with polygon–vector (defined in p.11) $(0, f_4, f_5, f_6)$, where $12 = f_4 + f_5 + f_6$. In fact, the polygon–vector can only be $(0, 2, 8, 2)$, $(0, 3, 6, 3)$ or $(0, 4, 4, 4)$ and so the combinatorial type of the regular dodecahedron does not arise in this setting.

Our classification of the combinatorial types of $\operatorname{span}(A)$ depends on the type–vector (defined in p. 25) and number and adjacency of the hexagonal facets in $\operatorname{span}(A)$ (knowing the polygon–vector is not enough!). The type–vector can be $(2, 2, 2)$, $(3, 2, 1)$, $(3, 3, 0)$, $(4, 1, 1)$ or $(4, 2, 0)$, up to a permutation. **Six** different classes exist, although just five classes have been announced in [18].

This paper arose from reading [18] and also [10]. A classification of tropical tetrahedra (with a less restrictive definition than ours) has been announced in [18], containing **five** combinatorial types. In [18], a very brief general description, plus five matrices and five pictures are provided in only half a page. There, the f–vector is claimed to be $(20, 30, 12)$ for every such $\operatorname{span}(A)$. Notice that this is the f–vector of the **regular dodecahedron** $\mathcal{D}$. We have two concerns about the five–item list in [18]. First, no description of each particular item is given. Which are the polygon–vectors, i.e., which polygons appear as facets and how many of each, for each item? Just looking at [18], we cannot tell. Moreover, we cannot answer another natural question such as: does any item in the list have the combinatorial type of $\mathcal{D}$? These questions remain unanswered after looking at the web page by the same authors, http://wwwopt.mathematik.tu-darmstadt.de/$\sim$kulas/Polytrope.html, containing additional information. Our second concern is that the list in [18] misses one class. It misses class 3, having type–vector $(3, 2, 1)$ and two adjacent hexagons (the polygon–vector is $(0, 2, 8, 2)$); see examples 20. This class is very important, since it shows that $\operatorname{span}(A)$ is not vertex–transitive, in general.

The new material is found in section 3. The previous sections are introductory. We have tried to make the paper self–contained. We have computed a lot of examples, thoroughly, for the benefit of the reader.

An earlier extended version of this paper was uploaded on 18/05/2012 in arXiv 1205.4162 with the title "Characterizing the convexity of the n–dimensional tropical simplex and the six convex classes in $\mathbb{R}^3$".

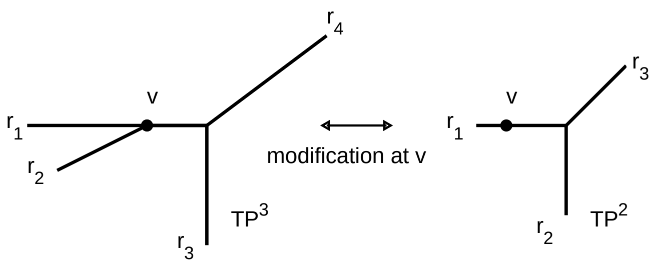

Figure 1: Modification of a tropical line at v: going from $\mathbb{TP}^3$ to $\mathbb{TP}^2$ and back.

2 Background and notations

We will work over $\mathbb{T} := (\mathbb{R}, \oplus, \odot)$, where $\oplus = \max$ is tropical addition and $\odot = +$ is tropical multiplication. For $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, 2, \ldots, n\}$. For $n, m \in \mathbb{N}$, $\mathbb{R}^{m \times n}$ denotes the set of real matrices having m rows and n columns. Define tropical sum and product of matrices following the same rules of classical linear algebra, but replacing addition (multiplication) by tropical addition (multiplication). We will never use classical sum or multiplication of matrices, in this note. $A \odot B$ will be written AB, for simplicity, for matrices A, B.

The **tropical determinant** (also called tropical permanent) of $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is defined as

$$|A|_{\mathrm{trop}} = \max_{\sigma \in S_n} \{a_{1\sigma(1)} + a_{2\sigma(2)} + \cdots + a_{n\sigma(n)}\},$$

where S_n denotes the permutation group in n symbols. The matrix A is **tropically singular** if this *maximum is attained, at least, twice*. Otherwise, A is **tropically regular**.

The **projective tropical** $n-1$**–dimensional space**, denoted $\mathbb{TP}^{n-1}$ is the quotient $\mathbb{R}^n / \sim$, where $(a_1, \ldots, a_n) \sim (b_1, \ldots, b_n)$ if and only if $(a_1, \ldots, a_n) = (\lambda + b_1, \ldots, \lambda + b_n)$, for some $\lambda \in \mathbb{R}$. The class of $(a_1, \ldots, a_n)$ will be denoted $[a_1, \ldots, a_n]$. Let $X_1, \ldots, X_n$ denote the coordinates on $\mathbb{TP}^{n-1}$. For any point in $\mathbb{TP}^{n-1}$, we can choose a representative $(a_1, \ldots, a_{n-1}, 0)$, in which case we say that

we **work in** $X_n = 0$. This allows us to identify $\mathbb{TP}^{n-1}$ with $\mathbb{R}^{n-1}$. We will make this identification throughout the paper.

From now on, coordinates of points will be written in columns.

Linear spaces in $\mathbb{TP}^2$ and $\mathbb{TP}^3$. It is well–known that tropical lines, planes, hyperplanes are piece–wise linear and piece–wise convex objects; see [1, 5, 6, 9, 13, 14, 16, 17, 23, 27, 32]. Let us call the pieces (linear convex sets) **building blocks**. For instance, the building blocks of tropical lines are unbounded rays and, sometimes, segments. And the building blocks of tropical planes are unbounded quadrants. These building blocks have rational slopes, i.e., orthogonal vectors to them can be chosen to have integer coordinates.

Fix $a_j \in \mathbb{R}$. The tropical linear form

$$a_1 \odot X_1 \oplus a_2 \odot X_2 \oplus a_3 \odot X_3 = \max\{a_1 + X_1, a_2 + X_2, a_3 + X_3\} \tag{1}$$

defines a tropical line in $\mathbb{TP}^2$, denoted Π_a: it is the set of points $[x_1, x_2, x_3]^t$ in $\mathbb{TP}^2$ where the **maximum is attained twice, at least.** Notice that the maximum is attained three times at $[-a_1, -a_2, -a_3]^t$. This point is called the **vertex** of Π_a. Working in $X_3 = 0$ (i.e., identifying $\mathbb{TP}^2$ with $\mathbb{R}^2$ in a certain way), the graphical representation of Π_a is the union of three rays r_1, r_2, r_3 meeting at point $(a_3 - a_1, a_3 - a_2)$. The ray r_j points towards the negative X_j direction, for $j = 1, 2$, and r_3 points towards the positive $X_1 = X_2$ direction; see right–hand–side of figure 1.

The tropical linear form

$$a_1 \odot X_1 \oplus a_2 \odot X_2 \oplus a_3 \odot X_3 \oplus a_4 \odot X_4 = \max\{a_1 + X_1, a_2 + X_2, a_3 + X_3, a_4 + X_4\} \tag{2}$$

defines a tropical plane Π_a in $\mathbb{TP}^3$: Π_a is the set of points $[x_1, x_2, x_3, x_4]^t$ in $\mathbb{TP}^3$ where the **maximum is attained twice, at least.** Notice that the maximum is attained four times at $[-a_1, -a_2, -a_3, -a_4]^t$. This special point, denoted v^{Π_a}, is called the **vertex** of Π_a. Working in $X_4 = 0$, we get Π_a as a subset of $\mathbb{R}^3$: it is the union of six 2–dimensional quadrants; see figure 2.

If $a_4 = 0$ then, the quadrants of Π_a are:

- $X_1 = a_1$, $X_2 \le a_2$, $X_3 \le a_3$, denoted Q_1,
- $X_1 \le a_1$, $X_2 = a_2$, $X_3 \le a_3$, denoted Q_2,
- $X_1 \le a_1$, $X_2 \le a_2$, $X_3 = a_3$, denoted Q_3,
- $a_1 \le X_1 = X_2$, $X_3 \le X_1 = X_2$, denoted Q_{12},
- $a_2 \le X_2 = X_3$, $X_1 \le X_2 = X_3$, denoted Q_{23},
- $a_3 \le X_3 = X_1$, $X_2 \le X_3 = X_1$, denoted Q_{31}.

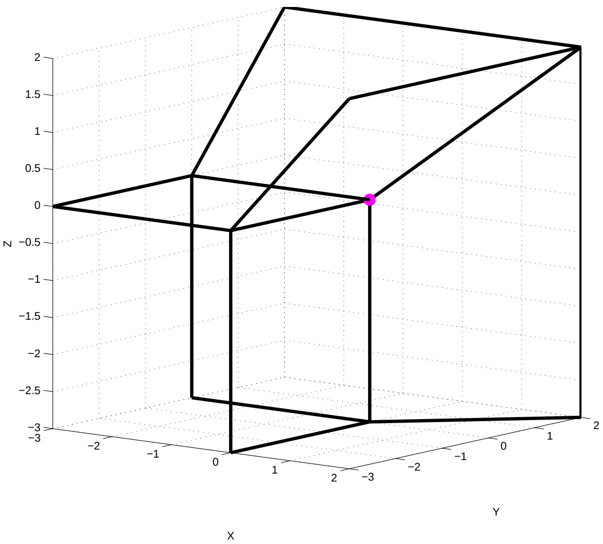

Figure 2: A tropical plane is the union of six closed quadrants.

It is well–known that three tropically non–collinear points p, q, r in $\mathbb{TP}^3$ determine a unique tropical plane, denoted $L(p, q, r)$, which passes through them; see [27]. The vertex of $L(p, q, r)$ is computed from the coordinates of p, q, r by the *tropical Cramer's rule*; see [27, 31]. Inside $L(p, q, r)$, the points p, q, r can be arranged as follows:

- all three in one quadrant, or
- two in one quadrant, the third one in another quadrant, or
- each one in a different quadrant (this is the generic case).

We have already explained that a tropical line in $\mathbb{TP}^2$ is a **tripod**: it has one vertex and three rays. Now, a tropical line in $\mathbb{TP}^3$ is **not homeomorphic** to a tripod (and this is a crucial difference between classical and tropical mathematics: lines in $\mathbb{P}^n$ and $\mathbb{P}^m$ are homeomorphic, if $n \neq m$). A generic tropical line L in $\mathbb{TP}^3$ has two vertices. Its building blocks are one edge e (i.e., a segment of finite length) and four unbounded rays r_1, r_2, r_3, r_4. Ray r_j points in the j–th negative coordinate direction, for $j = 1, 2, 3$ and r_4 points towards the positive direction of $X_1 = X_2 = X_3$; see figure 3 (unfortunately, our planar graphical representation of tropical lines in $\mathbb{TP}^3$ cannot show angles properly).

Any line L in $\mathbb{TP}^3$ belongs to one of the following types:

$$[12, 34], \qquad [13, 24], \qquad [14, 23], \qquad [1234]. \tag{3}$$

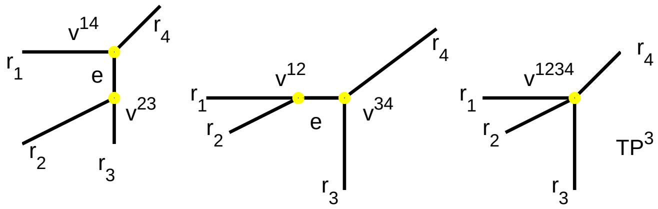

Figure 3: Graphical representation of some tropical lines in $\mathbb{TP}^3$: types $[14,23]$ on the left, $[12,34]$ center, tetrapod, on the right. These objects lie in $\mathbb{R}^3$; they are non–planar. The ray r_4 points towards the positive direction $X_1 = X_2 = X_3$.

Let us explain further. For type $[ij,kl]$, the vertices of L will be named v^{ij} and v^{kl} and the edge e joins them. Moreover, e, r_i and r_j meet at v^{ij}; same for e, r_k, r_l and v^{kl}. If the edge e does not exist in a given line L, then the two vertices of L coincide and the type of L is $[1234]$. In this case we say that L is a **tetrapod**. Types of tropical lines in $\mathbb{TP}^3$ can be written in various ways: for example, $[12,34] = [21,34] = [21,43] = [34,12]$.

Let us see how do types arise. It is well–known that two different points p,q in $\mathbb{TP}^3$ determine a unique tropical line, denoted $L(p,q)$, which passes through them. Following [30], the type and vertices of $L(p,q)$ are computed as follows. For $1 \le i < j \le 4$, consider the 2×2 tropical minors:

$$m_{ij} = \begin{vmatrix} p_i & q_i \\ p_j & q_j \end{vmatrix}_{\text{trop}}. \tag{4}$$

These minors satisfy the **tropical Plücker relation**, i.e., the following maximum is attained twice, at least:

$$\max\{m_{12} + m_{34}, m_{13} + m_{24}, m_{14} + m_{23}\}. \tag{5}$$

Then

- type $[12,34]$ arises when $m_{12} + m_{34} < m_{13} + m_{24} = m_{14} + m_{23}$,

- type $[13, 24]$ arises when $m_{13} + m_{24} < m_{12} + m_{34} = m_{14} + m_{23}$,
- type $[14, 23]$ arises when $m_{14} + m_{23} < m_{12} + m_{34} = m_{13} + m_{24}$,
- type $[1234]$ arises when $m_{12} + m_{34} = m_{13} + m_{24} = m_{14} + m_{23}$.

Now, a natural question is to determine the line $L(p, q)$ (and its type), for two given points $p \neq q$. In order to do so, it is enough to determine the vertex or vertices of $L(p, q)$. A point x belongs to $L(p, q)$ if and only if:

$$\operatorname{rk} \begin{bmatrix} p_1 & q_1 & x_1 \\ p_2 & q_2 & x_2 \\ p_3 & q_3 & x_3 \\ p_4 & q_4 & x_4 \end{bmatrix}_{\mathrm{trop}} = 2. \tag{6}$$

This **tropical rank** condition means that the value of each of the following 3×3 tropical minors is attained twice, at least (see [11] for tropical rank issues):

$$\begin{vmatrix} p_2 & q_2 & x_2 \\ p_3 & q_3 & x_3 \\ p_4 & q_4 & x_4 \end{vmatrix}_{\mathrm{trop}} = \max\{x_2 + m_{34}, x_3 + m_{24}, x_4 + m_{23}\} \tag{7}$$

$$\begin{vmatrix} p_1 & q_1 & x_1 \\ p_3 & q_3 & x_3 \\ p_4 & q_4 & x_4 \end{vmatrix}_{\mathrm{trop}} = \max\{x_1 + m_{34}, x_3 + m_{14}, x_4 + m_{13}\} \tag{8}$$

$$\begin{vmatrix} p_1 & q_1 & x_1 \\ p_2 & q_2 & x_2 \\ p_4 & q_4 & x_4 \end{vmatrix}_{\mathrm{trop}} = \max\{x_1 + m_{24}, x_2 + m_{14}, x_4 + m_{12}\} \tag{9}$$

$$\begin{vmatrix} p_1 & q_1 & x_1 \\ p_2 & q_2 & x_2 \\ p_3 & q_3 & x_3 \end{vmatrix}_{\mathrm{trop}} = \max\{x_1 + m_{23}, x_2 + m_{13}, x_3 + m_{12}.\} \tag{10}$$

Now, for any u positive, big enough real number, it is obvious that the points $x_j(u)$ below make the maxima attained, at least, twice, in expressions (7), (8), (9), (10), respectively:

$$x_1(u) = [-u, -m_{34}, -m_{24}, -m_{23}]^t \quad x_2(u) = [-m_{34}, -u, -m_{14}, -m_{13}]^t, \tag{11}$$

$$x_3(u) = [-m_{24}, -m_{14}, -u, -m_{12}]^t, \quad x_4(u) = [-m_{23}, -m_{13}, -m_{12}, -u]^t \tag{12}$$

And as u goes from zero to $+\infty$, the point $x_j(u)$ moves along a ray r_j.

Say the type of $L(p,q)$ is $[12,34]$. Then a value of u can be determined so that $x_1(u) = x_2(u) = v^{12}$ (resp. $x_3(u) = x_4(u) = v^{34}$), obtaining the following vertices for $L(p,q)$:

$$v^{12} = [m_{13} - m_{23} - m_{34}, -m_{34}, -m_{24}, -m_{23}]^t. \tag{13}$$

$$v^{34} = [-m_{24}, -m_{14}, m_{13} - m_{12} - m_{14}, -m_{12}]^t. \tag{14}$$

Say the type of $L(p,q)$ is $[13,24]$. Similar calculations yield the following vertices for $L(p,q)$:

$$v^{13} = [m_{24}, -m_{14}, -m_{24} - m_{14} + m_{34}, -m_{12}]^t. \tag{15}$$

$$v^{24} = [-m_{23}, -m_{13}, -m_{12}, -m_{13} - m_{12} + m_{14}]^t. \tag{16}$$

Say the type of $L(p,q)$ is $[1234]$. Then $x_1(u) = x_2(u) = x_3(u) = x_4(u)$ for some u, providing the vertex

$$v^{1234} = [m_{13} + m_{14} - m_{34}, m_{12}, m_{13}, m_{14}]^t. \tag{17}$$

Computations are similar for type $[14,23]$.

Notice that the coordinates of the vertices of $L(p,q)$ depend on the type of $L(p,q)$.

Lines as tropical algebraic varieties. An integer vector is *primitive* if its coordinates are relatively prime. It is well–known that a *tropical algebraic variety* V satisfies the **balance condition** (see [13, 17, 24, 27]) at each point $p \in V$: this means that

$$w(p)_1 + \cdots + w(p)_{s(p)} = 0, \tag{18}$$

where $s(p) \geq 2$ and $w(p)_1, \ldots, w(p)_{s(p)}$ are all the *weighted primitive vectors* which are outward normal to the different $s(p)$ building blocks of V meeting at p.

In p. 5 we have seen that lines in $\mathbb{TP}^2$ are *not homeomorphic* to lines in $\mathbb{TP}^3$. But, as tropical varieties, lines must be all the same (i.e., they must be equivalent, in some way), regardless of the embedding dimension. The key concept to get such an equivalence relation is called **modification**; see [23] for details. We will briefly explain modifications only for lines. Suppose $L \subset \mathbb{TP}^3$ is a tropical line in which rays r and r' meet at vertex $v \in L$ (and either an edge e or two more rays meet also at v). The balance condition for L holds at v. Roughly speaking, **a modification of L at the point** v consists in contracting one ray (either r or r') down to the point v obtaining something, denoted $\overline{L}$, as a result. $\overline{L}$ must satisfy the balance condition, so at the same time of contracting one ray, a straightening of the direction of the second ray is necessary. The resulting object $\overline{L}$, viewed in $\mathbb{TP}^2$, satisfies the balance condition at every point, also at v. $\overline{L}$ is is a tropical line in $\mathbb{TP}^2$, called a modification of L at v. The inverse procedure, i.e., passing from $\overline{L}$ to L

is also called a modification; see figure 1, with $r = r_1$ and $r' = r_2$ or $r = r_2$ and $r' = r_1$.

The **tropical distance** between two points $p = [p_1, p_2, p_3, 0]^t$ and $q = [q_1, q_2, q_3, 0]^t$ is

$$\operatorname{tdist}(p, q) := \max\{|p_i - q_i| \ : \ i = 1, 2, 3\}. \tag{19}$$

Notice that if three coordinates in p and q coincide, then the tropical and Euclidean distances between p and q coincide.

Let $n, m \in \mathbb{N}$ and $a_1, \ldots, a_n$ be points in $\mathbb{TP}^m$. The **tropical span** of $a_1 \ldots, a_n$ is

$$\operatorname{span}(a_1, \ldots, a_n) := \{(\lambda_1 + a_1) \oplus \cdots \oplus (\lambda_n + a_n) \in \mathbb{TP}^m : \lambda_1, \ldots, \lambda_n \in \mathbb{R}\}, \tag{20}$$

where maxima are computed coordinate–wise. If $n = 2, 3$ or 4, we speak of **tropical segment**, **tropical triangle** or **tropical tetrahedron**.

Assume $m = n - 1$ and let us write the coordinates of the a_j as the columns of a matrix A. In order to view $\operatorname{span}(A)$ inside $\mathbb{R}^{n-1}$, we must use the matrix $A_0 = (\alpha_{ij})$, where

$$\alpha_{ij} = a_{ij} - a_{nj}. \tag{21}$$

By (20), $x = [x_1, \ldots, x_{n-1}, 0]^t \in \operatorname{span}(A)$ if and only if there exist $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ such that

$$0 = \max_{k \in [m]} \lambda_k, \quad x_j = \max_{k \in [m]} \lambda_k + \alpha_{kj}, \quad j \in [n-1]. \tag{22}$$

Kleene stars. Consider $A \in \mathbb{R}^{n \times n}$. By definition (see [7, 28, 8, 29]), A is *a Kleene star* if A is null–diagonal and idempotent, tropically; in symbols: $\operatorname{diag}(A) = 0$ and $A = A^2$. Notice that, if every diagonal entry of $A = (a_{ij})$ vanishes, then $A \leq A^2$, because for each $i, j \in [n]$, we have $(A^2)_{ij} = \max_{k \in [n]} a_{ik} + a_{kj} \geq a_{ij}$. Therefore, being a Kleene star is characterized by the following n equalities and $\binom{n}{2} + \binom{n}{3}$linear inequalities:

$$a_{ii} = 0, \quad a_{ik} + a_{kj} \leq a_{ij}, \quad i, j, k \in [n], \quad \operatorname{card}\{i, j, k\} \geq 2. \tag{23}$$

3 Convex maximal tropical tetrahedra in $\mathbb{TP}^3$

Assume A is an order 4 matrix. The objective of this paper is to find all the possible combinatorial types of the tropical tetrahedron $\operatorname{span}(A)$, when $\operatorname{span}(A)$ is convex and maximal (maximality to be defined in p. 17). By [26, 28], $\operatorname{span}(A)$ is convex if and only if A is a Kleene star, and then $\operatorname{span}(A)$ is determined by the following 12 inequalities:

$$\begin{array}{l|l} a_{14} \leq X_1 \leq -a_{41} & a_{12} \leq X_1 - X_2 \leq -a_{21} \\ a_{24} \leq X_2 \leq -a_{42} & a_{23} \leq X_2 - X_3 \leq -a_{32} \\ a_{34} \leq X_3 \leq -a_{43} & a_{31} \leq X_3 - X_1 \leq -a_{13}. \end{array} \tag{24}$$

From now on, we will assume that A is an order 4 Kleene star. For $i, j \in [4]$, $i \neq j$, let $L_{ij} := L(\mathrm{col}(A, i), \mathrm{col}(A, j))$.

3.1 The type of the tropical line L_{ij}

Here is an application of the balance condition in $\mathbb{TP}^3$ identified with $\mathbb{R}^3$. Let e_1, e_2, e_3 denote the canonical vectors, $e_{ij} := e_i + e_j$, for $i \neq j$ and $e_{123} := e_1 + e_2 + e_3$. Suppose L is a tropical line, not a tetrapod. Then the direction of the edge of L is e_{ij} if and only if the type of L is $[ij, k4]$, with $\{i, j, k\} = [3]$.

Theorem 1. *Assume A is an order 4 Kleene star. Let $\{i, j, k, l\} = [4]$ with $i < j$. Then the* ***type*** *of the tropical line L_{ij} is either $[ik, jl]$ or $[il, jk]$ or else L_{ij} is a tetrapod (easy to remember: i* ***and*** *j* ***are separated by the comma****, unless L_{ij} has just one vertex).*

Proof. Without lost of generality, assume that $i = 1$, $j = 2$, so that $\{k, l\} = \{3, 4\}$. Write $p = \mathrm{col}(A, 1)$ and $q = \mathrm{col}(A, 2)$ and compute the tropical minors m_{ij}'s as in expression (4). Write $M = |A(34; 12)|_{\mathrm{trop}}$. Then

$$m_{12} + m_{34} = M, \; m_{13} + m_{24} = a_{32} + a_{41}, \; m_{14} + m_{23} = a_{31} + a_{42}.$$

The value M is attained at the main (resp. secondary) (resp. both) diagonal(s) if and only if the type of line L_{12} is $[13, 24]$ (resp. $[14, 23]$) (resp. $[1234]$). This proves the statement. $\square$

3.2 Generators and extremals

From now on, we assume that A is an order 4 Kleene star such that **the columns of A represent four tropically non–coplanar points in $\mathbb{TP}^3$**: this is our **hypothesis 1**.

In [18], a tropical tetrahedron is defined as the tropical tropical span of **four points which are not contained in the boundary of a tropical halfspace**. If four points are not tropically coplanar, then they are not contained in the boundary of a tropical halfspace, but the converse is not true. Therefore, our hypothesis 1 is more restrictive (and more natural, in our opinion) than the hypothesis in [18]. In particular, if (within a smaller set of matrices) we find six combinatorial classes (see below, p. 30), then there should be at least six classes in [18], but only five classes are shown there.

Our aim is to study $\mathrm{span}(A)$ as a convex body in 3–dimensional space. It turns out that $\mathrm{span}(A)$ is not regular, i.e., its facets are irregular polygons. However, the facets of $\mathrm{span}(A)$ are nice enough, because they are contained in classical planes of equations $X_i = \mathrm{cnst}$, $X_j - X_k = \mathrm{cnst}$, $i, j, k \in [3]$. In other words, the edges of $\mathrm{span}(A)$ have directions e_i, e_{jk}, or e_{123}. Such polyhedra are called **alcoved polyhedra**; see [18, 19, 20, 33].

We work in $\mathbb{TP}^3$, identified with $\mathbb{R}^3$, using the matrix $A_0 = (\alpha_{ij})$ defined in (2). The points represented by the columns of A_0 will be called **generators**. Taking generators three by three, they yield four additional points (the four vertices of the four corresponding tropical planes) and taking generators two by two, they yield, at most, twelve more points (the vertices of six tropical lines). These new points will be called **extremal generated points**. Generators and extremal generated points are called **extremals** of $\operatorname{span}(A)$. In the generic case, we get a total of $20 = 4 + 4 + 12$ *different extremals* and then we say that A is **maximal in extremals**. Notice that $20 = \binom{2n-2}{n-1}$, for $n = 4$, which agrees with [10, 15, 18]. The number and computation of extremals have been studied in [2, 3], in a more general setting (where classical convexity is not assumed).

The following **color code for figures** will be used: blue for generators, magenta for vertices of tropical planes and yellow for vertices of tropical lines. Two adjacent red segments should be glued together, after cutting and folding. Dashed segments must be mountain–folded, dotted segments must be valley–folded.

3.3 Some tropical triangles in $\mathbb{TP}^3$, their matrices and vertex configurations

Now, we must sidetrack to discuss what some tropical triangles in $\mathbb{TP}^3$ are like. The columns of A_0, taken three by three, determine four tropical triangles in $\mathbb{TP}^3$. Tropical triangles in $\mathbb{TP}^3$ can be easily understood. Planar tropical triangles have been studied in [4, 10, 18, 25]. In general, tropical triangles are compact but not convex.

Let B be a 4×3 real matrix obtained by deleting one column in A. We want to describe the tropical triangle $\operatorname{span}(B)$. The points represented by the columns of B_0 will be called the **generators** of $\operatorname{span}(B)$. They determine one tropical plane Π^B and three tropical lines: L_{12}, L_{23} and L_{31}, where L_{ij} denotes $L(\operatorname{col}(B,i), \operatorname{col}(B,j))$. Thus, **additional extremals** arise: the vertex of the plane Π^B and the vertices of the lines. In the generic case, we get a total of $10 = 3 + 1 + 2 \times 3$ different extremals in $\operatorname{span}(B)$ and we say that B is **maximal in extremals**.

The **f–vector** of $\operatorname{span}(B)$ is (v, e, f), where v (resp. e) (resp. f) is the number of extremals (resp. edges) (facets) of $\operatorname{span}(B)$. The *Euler characteristic* of $\operatorname{span}(B)$ is $v - e + f = 1$. The **polygon–vector** of $\operatorname{span}(B)$ is (f_3, f_4, f_5, f_6), where $f_m \geq 0$ is the number of convex m–gons occurring as facets and $f = f_3 + f_4 + f_5 + f_6$.

We will only consider the maximal case, i.e., $v = 10$. Inside Π^B, the generators of $\operatorname{span}(B)$ can sit as follows:

- all three in one quadrant (therefore, the triangle $\operatorname{span}(B)$ is planar), or
- two in one quadrant, the third one in another quadrant, or

- each generator in a different quadrant (this is the generic case).

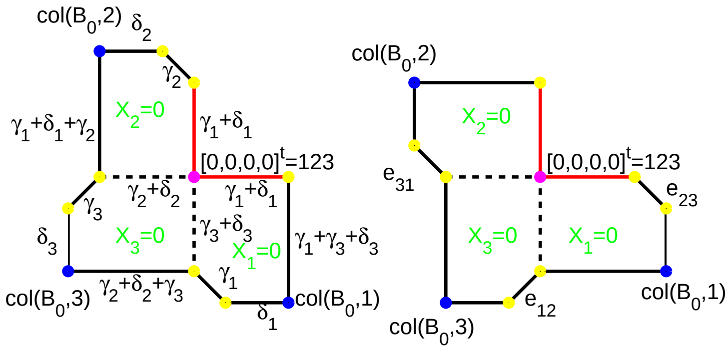

Figure 4: Cut–and–fold models for a tropical triangle $\mathrm{span}(B)$ made up of three pentagons. Point v^{Π^B} is of type (5.5.5): left and right models. In the figure, the coordinates of v^{Π^B} are $[0,0,0,0]^t$.

We need only study the generic case (i.e., $f = 3$), since the second case (i.e., $f = 2$) is a degeneration of it, and the first case (i.e., $f = 1$) has already been studied in [4, 10, 18, 25]. In the generic case, $\mathrm{span}(B)$ is the union of three classical convex m–gons, one m–gon contained in each quadrant, with $m = 3, 4, 5, 6$. How are the ten extremals of $\mathrm{span}(B)$ distributed among the quadrants of Π^B? By genericity, each generator lies on a different quadrant and only three quadrants are involved. Also, the vertex of Π^B is common to all three quadrants. And, for each two quadrants (out of three), one additional extremal point lies on their intersection. This leaves out three extremals. How are these three distributed among the three quadrants involved? They cannot lie all on just one quadrant, because this would yield more than six extremal points (generators or additional) on one quadrant, but our polygons have six vertices, at most. So, the three remanent extremals can be arranged as follows:

- one in each quadrant, or
- two in one quadrant and one in another quadrant.

In the first case, the tropical triangle $\mathrm{span}(B)$ is the union of **three pentagons**, i.e., the polygon–vector of $\mathrm{span}(B)$ is $(0, 0, 3, 0)$. This situation is also expressed

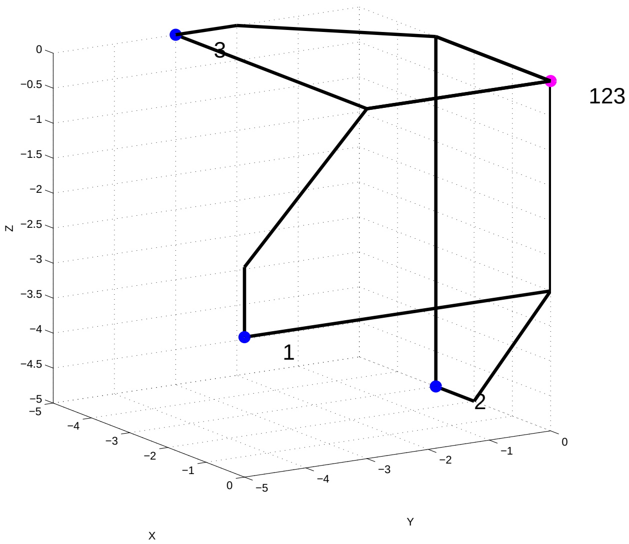

Figure 5: 3–dimensional model for matrix (25) with $\gamma_j = 2$, $\delta_j = 1$, $j = 1, 2, 3$ left.

by saying that the **configuration at the point** v^{Π^B} is $(5.5.5)$. There is a left version and a right version of configuration $(5.5.5)$; see cut–and–fold models in figure 4, and the corresponding folded models in figures 5 and 6.

In this case, **what is** B_0 **like?** Let us assume that $v^{\Pi^B} = [0,0,0,0]^t$ and the deleted column in A is the fourth one. Then

$$B_0 = \begin{bmatrix} 0 & -\gamma_2 - \delta_2 & -\gamma_3 - \gamma_2 - \delta_2 \\ -\gamma_1 - \gamma_3 - \delta_3 & 0 & -\gamma_3 - \delta_3 \\ -\gamma_1 - \delta_1 & -\gamma_2 - \gamma_1 - \delta_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{left model}, \quad (25)$$

and

$$B_0 = \begin{bmatrix} 0 & -\gamma_2 - \gamma_3 - \delta_3 & -\gamma_3 - \delta_3 \\ -\gamma_1 - \delta_1 & 0 & -\gamma_3 - \gamma_1 - \delta_1 \\ -\gamma_1 - \gamma_2 - \delta_2 & -\gamma_2 - \delta_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{right model}, \quad (26)$$

for some parameters $\gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, \delta_3 > 0$.

In the second case, the tropical triangle $\mathrm{span}(B)$ is the union of **a quadrangle, a pentagon and a hexagon**, i.e., the polygon–vector of $\mathrm{span}(B)$ is $(0,1,1,1)$. We say that the **configuration at the point** v^{Π^B} is $(4.5.6)$ if the quadrangle is contained in the plane $X_1 = \mathrm{cnst}$, the pentagon in $X_2 = \mathrm{cnst}$, and the hexagon in $X_3 = \mathrm{cnst}$.

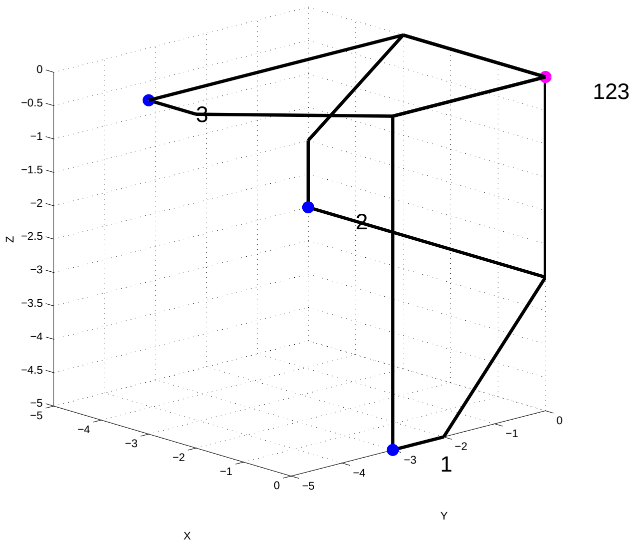

Figure 6: 3–dimensional model for matrix (26) with $\gamma_j = 2$, $\delta_j = 1$, $j = 1, 2, 3$ right.

Similarly, we have configurations $(p.q.r)$ for any $\{p, q, r\} = \{4, 5, 6\}$. See figures 7 and 8 for a picture of some of these configurations.

What is B_0 like? Let us assume $v^{\Pi^B} = [0, 0, 0, 0]^t$ and the deleted column in A is the last one. After a change of variables we have

$$B_0 = \begin{bmatrix} 0 & -\gamma_2 - \gamma_3 - \delta_3 & -\gamma_3 - \delta_3 \\ -\delta_1 & 0 & -\delta_1 - \gamma_3 \\ -\gamma_2 - \delta_2 + \lambda_2 & -\gamma_2 - \delta_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{configuration } (4.6.5), \tag{27}$$

for some parameters $\delta_1, \gamma_2, \delta_2, \lambda_2, \gamma_3, \delta_3 > 0$, with $\gamma_2 + \delta_2 - \lambda_2 > 0$, and

$$B_0 = \begin{bmatrix} 0 & -\delta_2 & -\delta_2 - \gamma_3 \\ -\gamma_1 - \gamma_3 - \delta_3 & 0 & -\gamma_3 - \delta_3 \\ -\gamma_1 - \delta_1 & -\gamma_1 - \delta_1 + \lambda_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{configuration } (6.4.5), \tag{28}$$

for some parameters $\gamma_1, \delta_1, \lambda_1, \delta_2, \gamma_3, \delta_3 > 0$, with $\gamma_1 + \delta_1 - \lambda_1 > 0$. We get similar expressions for B_0, for other configurations $(p.q.r)$.

In summary, if B is generic and maximal in extremals (i.e., $f = 3$ and $v = 10$), then the possible configurations at the vertex of $\operatorname{span}(B)$ are:

1. $(5.5.5)$ left,

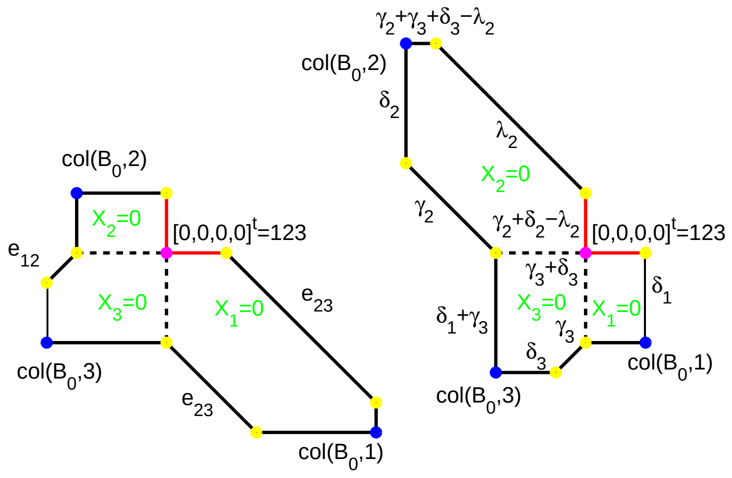

Figure 7: Cut–and–fold models for a tropical triangle $\mathrm{span}(B)$ made up of a quadrangle, a pentagon and a hexagon. The configuration of v^{Π^B} is $(6.4.5)$ for the left figure, and $(4.6.5)$ for the right figure.

2. $(5.5.5)$ right,
3. $(p.q.r)$, with $\{p, q, r\} = \{4, 5, 6\}$.

In the previous examples, the generators of $\mathrm{span}(B)$ lie on Q_1, Q_2 and Q_3, introduced in p. 4. These quadrants are orthogonal to each other, and so we say that the **angle–vector** at the vertex of $\mathrm{span}(B)$ is $\langle 90, 90, 90\rangle$, in degrees. But if some generator lies on quadrant Q_{ij}, other angle–vectors will occur. Indeed, the intersection of two quadrants can be parallel to vector e_j, for $j = 1, 2, 3$, or to the vector e_{123}. Thus, by elementary geometry, the following angles occur:

$$\overline{\alpha} := \angle(e_j, e_{123}) = \arccos(\sqrt{1/3}) \simeq 54^\circ 44'$$

and its supplementary

$$\alpha \simeq 125^\circ 16'. \tag{29}$$

So, the angle–vector $\langle 90, \alpha, \alpha\rangle$ is also possible at the vertex of a tropical triangle $\mathrm{span}(B)$.

Above, we have considered the **angle–vector at the vertex of a tropical triangle**. We can also speak of the **angle–vector of a polygon**. All the polygons that we

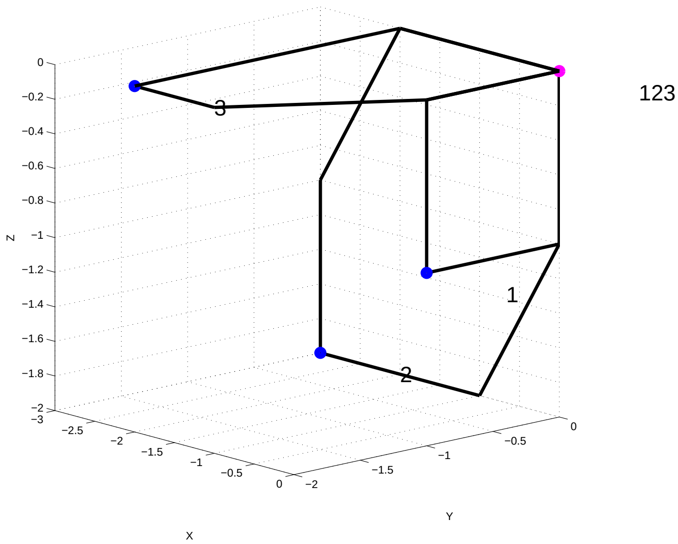

Figure 8: 3–dimensional model for matrix (27) with $\delta_1 = \gamma_2 = \delta_2 = \lambda_2 = \gamma_3 = \delta_3 = 1$. The configuration is $(4.6.5)$.

will encounter below are planar tropical triangles. It is well–known that some tropical planar triangles are classical hexagons; see [4]. The edges of these hexagons have directions e_1, e_2 and e_{12}, the angles occurring there being

$$\angle(e_j, e_{12}) = \arccos(\sqrt{1/2}) = 45^\circ, \quad 180 - 45 = 135^\circ, \quad 90^\circ, \tag{30}$$

whence $\langle 90, 135, 135, 90, 135, 135\rangle$ is the angle–vector of such a hexagon; see figure 9, upper left. Pentagons, quadrangles or triangles in the same figure are obtained from the given hexagon, when one, two or three edges have collapsed. The corresponding angle–vectors are thus easily deduced. Later on, we will also encounter the following angles

$$\overline{\beta} := \angle(e_{jk}, e_{123}) = \arccos(\sqrt{2/3}) \simeq 35^\circ 16'$$

and its supplementary

$$\beta \simeq 144^\circ 44'. \tag{31}$$

3.4 Maximality and near–miss Johnson solids

Let us return to discuss convex tropical tetrahedra. Since each tropical triangle in $\mathbb{TP}^3$ is made up of, at most, 3 m–gons, for $m = 3, 4, 5, 6$, then $\operatorname{span}(A)$ may have

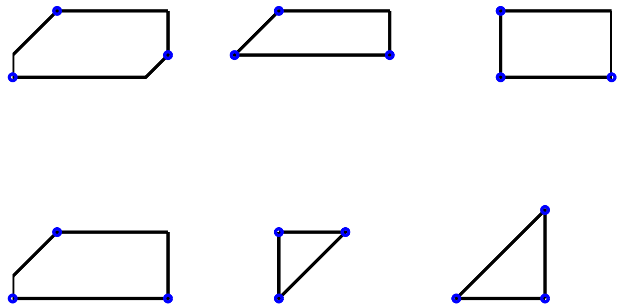

Figure 9: The planar case: some tropical triangles in $\mathbb{TP}^2$ defined by tropically idempotent 3×3 matrices. A hexagon is found in the upper left corner. The rest of the figures are obtained by letting some edges in such a hexagon collapse. Generators are marked in blue.

up to $4 \times 3 = 12$ facets (this agrees with proposition 5 in [18]). If this is the case, we will say that A is **maximal in facets**. If A is maximal in extremals (i.e., there are 20 such), then A is maximal in facets, because each facet contains, at most, six extremals. The converse is not true. In the sequel, we will just say that A is **maximal**, meaning maximal in extremals and facets. By [18], if A is maximal, then $\operatorname{span}(A)$ is **simple** or **trivalent**, i.e., three facets are concurrent at each extremal point.

By Euler's formula, with 20 vertices and 12 facets, $\operatorname{span}(A)$ must have 30 edges. But $(20, 30, 12)$ is precisely the f–vector of a *regular dodecahedron* $\mathcal{D}$ (the polygon–vector of $\mathcal{D}$ is, obviously, $(0, 0, 12, 0)$). $\mathcal{D}$ is one of the famous Platonic solids (or regular solids). A *Johnson solid* is a (less famous) convex polyhedron, each facet of which is a regular polygon (like Platonic solids) but it is not uniform, i.e., it is not vertex–transitive (unlike Platonic solids). Since 1969, it has been known that there are exactly 92 classes of Johnson solids. A convex polyhedron, each facet of which is near–regular is called a **near–miss Johnson solid.** This is a wide generalization of Johnson solids. Let us visualize one near–miss Johnson solid. Take a regular dodecahedron $\mathcal{D}$ and choose four equidistant vertices in $\mathcal{D}$. Now allow each chosen vertex to migrate to a neighboring facet. We obtain a new solid $\mathcal{D}'$, having the same f–vector and polygon–vector $(0, 4, 4, 4)$. $\mathcal{D}'$ is a near–miss Johnson solid. In lemma

18 we have two examples having the same combinatorial type as $\mathcal{D}'$. By tropicality and maximality, we will only deal with convex solids having $(20, 30, 12)$ as f–vector and polygon–vector $(0, f_4, f_5, f_6)$, with $12 = f_4 + f_5 + f_6$.

3.5 Oddly generated extremals

Fix an order 4 Kleene star A. We must first name, then compute the extremals in $\operatorname{span}(A)$. Here are some notations:

- Extremals of $\operatorname{span}(A)$ will be underlined. In particular, $\underline{1}$, $\underline{2}$, $\underline{3}$, $\underline{4}$ are the generators. It is just an abbreviation for $\operatorname{col}(A_0, j)$, $j \in [4]$.
- The vertex of the tropical plane $L(\underline{i}, \underline{j}, \underline{k})$ is denoted $\underline{ijk}$. Here the order of appearance if i, j, k is irrelevant.
- Let $\{i, j, k, l\} = [4]$.We say that the point $\underline{l}$ is **opposite** to the point $\underline{ijk}$ inside $\operatorname{span}(A)$.
- We say that the point $\underline{ijk}$ is 3–generated. The extremals $\underline{i}$ and $\underline{ijk}$ are **generated by an odd number of points**.

Lemma 2. *If A is an order 4 Kleene star, then the columns of $-A^t$ represent the points $\underline{234}$, $\underline{134}$, $\underline{124}$, $\underline{123}$.*

Proof. The coordinates of the points $\underline{ijk}$ are given by the tropical Cramer's rule; see [27, 31]. This means that the points $\underline{ijk}$ are given by the columns of the matrix $-\widehat{A}^t$. Here $\widehat{A} = (b_{ij})$ is the *tropical adjoint of* A, where b_{ij} equals the tropical determinant of the minor obtained by omitting the j–th row and i–column in A. An easy computation shows that $\widehat{A} = A$, for a Kleene star A. □

Let $s : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be the antipodal map, defined by $s(a) = -a$. The following lemma is easy to prove.

Lemma 3. *2 Let $A \in \mathbb{R}^{n\times n}$. The matrix A is a Kleene star if and only if A^t is. In such a case,* $\operatorname{span}(A^t) = s(\operatorname{span}(A))$. *In particular, if A is a Kleene star, then A is symmetric if and only if* $\operatorname{span}(A)$ *is symmetric with respect to the origin.* □

Set $n = 4$. The map s takes the extremal $\underline{i}$ in $\operatorname{span}(A)$ onto $\underline{jkl}$ in $\operatorname{span}(A^t)$. Thus, if A is a maximal Kleene star, the possible configurations at any oddly generated extremal point are summarized in p. 14. For instance, the configuration at point $\underline{4}$ is $(5.5.5)$ if three pentagons meet at $\underline{4}$. There are two possibilities, left and right, which are shown (unfolded) in figure 10. At point $\underline{4}$ we can also have configuration $(p.q.r)$, with $\{p, q, r\} = \{4, 5, 6\}$.

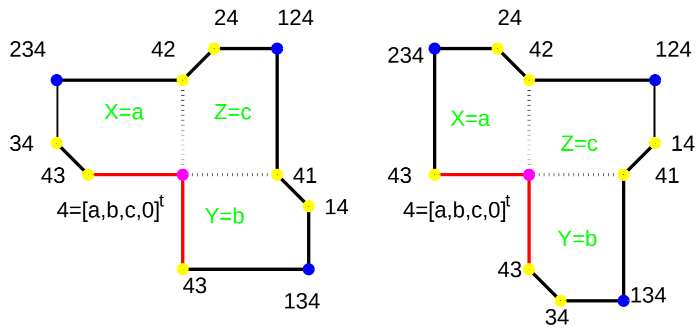

Figure 10: The configuration at point $\underline{4}$ is $(5.5.5)$, left or right. Here, the coordinates of point $\underline{4}$ are $[a, b, c, 0]^t$.

3.6 2–generated extremals, adiff and tropical distance

Given two points $p \neq q$ in $\mathbb{TP}^3$, denote by v, w the vertices of the tropical line $L(p, q)$. Pursuing maximality, we want to know *when* p, q, v, w, *are all different*, i.e., $\operatorname{card}\{p, q, v, w\} = 4$.

Lemma 4. *Suppose* p, q *are different points in* $\mathbb{TP}^3$ *and* v, w *are the vertices (perhaps,* $v = w$*) of the tropical line* $L(p, q)$*. Then* $\operatorname{card}\{p, q, v, w\} \leq 3$ *if and only if then there exists a tropical change of variables that gives* $p = [0, 0, a, b]^t$ *and* $q = [0, 0, 0, 0]^t$*, for some real numbers* a, b*.*

Proof. Suppose that $p = [0, 0, a, b]^t$ and $q = [0, 0, 0, 0]^t$. Write $M = a \oplus b$ and $a^+ = a \oplus 0$. Recall here the notation m_{ij} introduced in p. 6. By straightforward computations,

$$m_{12} + m_{34} = M, \quad m_{13} + m_{23} = m_{14} + m_{23} = a^+ + b^+.$$

Clearly, $M \leq a^+ + b^+$ and $L(p, q)$ is a tetrapod if and only if $M = a^+ + b^+$ or, equivalently, $ab \leq 0$. In this case, $\operatorname{card}\{p, q, v, w\} \leq 3$. Otherwise, $ab > 0$ and the type of $L(p, q)$ is $[12, 34]$, with

$$v^{12} = [-M, -M, -b^+, -a^+]^t, \quad v^{34} = [-b^+ - a^+, -b^+ - a^+, -b^+, -a^+]^t.$$

If we simplify the former expressions, in each particular case, we obtain

1. if $a = b$, then the vertices of $L(p,q)$ are p and q, so that $\operatorname{card}\{p,q,v,w\} = 2$,
2. if $a \neq b$, then the vertices of $L(p,q)$ are either p or q, but not both, and one more point, so that $\operatorname{card}\{p,q,v,w\} = 3$.

Suppose now that $p = [a,b,c,0]^t$ and $q = [0,0,0,0]^t$, with non–zero a,b,c and not $a = b = c$. The tropical line $L(p,q)$ has four rays, one in each negative coordinate direction e_j, $j \in [3]$, and one in the positive direction e_{123}. Moreover, at each point of $L(p,q)$ the balance condition holds. Since a,b,c are non–zero and they are different, we cannot go from q to p along $L(p,q)$ running through **only two** classical segments. Therefore, $L(p,q)$ has a bounded edge e (having direction e_{kl}, for some different $k,l \in [4]$) and $\operatorname{card}\{p,q,v,w\} = 4$.

$\square$

Here are more notations for a given order 4 matrix A. Choose $i \neq j$ in $[4]$; the vertices of the tropical line L_{ij} are $\underline{ij}$ and $\underline{ji}$ named so that $\underline{ij}$ is the **closest to** $\underline{i}$, with respect to tropical distance. Of course, $\underline{ji} = \underline{ij}$ if and only if L_{ij} is a tetrapod. We say that the extremals $\underline{ij}$ and $\underline{ji}$ are **2–generated**.

Now we introduce **adiffs**, which provide the tropical distance between some pairs of extremals. Let $i,j,k,l \in [n]$ with $k < l$ and $i < j$. By $A(kl;ij)$ we denote the 2×2 minor $\begin{bmatrix} a_{ki} & a_{kj} \\ a_{li} & a_{lj} \end{bmatrix}$. Write $\operatorname{adiff}_A(kl;ij)$ to denote

$$|a_{ki} + a_{lj} - a_{kj} - a_{li}|, \tag{32}$$

i.e., the **absolute value of the difference** of the items in the maximum below

$$|A(kl;ij)|_{\text{trop}} = \max\{a_{ki} + a_{lj}, a_{kj} + a_{li}\}. \tag{33}$$

We extend the notation as follows: $\operatorname{adiff}_A(lk;ij) = \operatorname{adiff}_A(kl;ji) = \operatorname{adiff}_A(lk;ji)$ are all equal to the already defined $\operatorname{adiff}_A(kl;ij)$, with $k < l$ and $i < j$.

The following properties are easy to check, for $i,j,k,l \in [n]$:

1. $\alpha_{ii} - \alpha_{ij} = \operatorname{adiff}_A(in;ij)$,
2. $\operatorname{adiff}_A(ij;kl) = \operatorname{adiff}_{A_0}(ij;kl)$.

We simply write $\operatorname{dist}(p,q)$, when the Euclidean and tropical distances between points p and q coincide.

Theorem 5. *Assume hypothesis 1 for an order 4 matrix A. Let $\{i,j,k,l\} = [4]$ with $i < j$. If the type of the tropical line L_{ij} is $[ik, jl]$, then*

$$\operatorname{dist}(\underline{i}, \underline{ij}) = \operatorname{adiff}_A(jl;ij), \quad \operatorname{dist}(\underline{j}, \underline{ji}) = \operatorname{adiff}_A(ik;ij),$$

$$\operatorname{tdist}(\underline{ij}, \underline{ji}) = \operatorname{adiff}_A(kl;ij).$$

Proof. Without lost of generality, assume that $i = 1$, $j = 2$, so that $\{k, l\} = \{3, 4\}$. Recall that the coordinates of the vertices of L_{12} depend on the type of L_{12}.

Say the type of L_{12} is $[13, 24]$; then $k = 3$, $l = 4$. Using formulas (15) and (16), the vertices of line L_{12} are

$$v^{13} = \begin{bmatrix} 0 \\ a_{41} - a_{42} \\ a_{31} \\ a_{41} \end{bmatrix} = \begin{bmatrix} -a_{41} \\ -a_{42} \\ a_{31} - a_{41} \\ 0 \end{bmatrix}, \quad v^{24} = \begin{bmatrix} a_{32} - a_{31} \\ 0 \\ a_{32} \\ a_{42} \end{bmatrix} = \begin{bmatrix} a_{32} - a_{31} - a_{42} \\ -a_{42} \\ a_{32} - a_{42} \\ 0 \end{bmatrix}. \tag{34}$$

The generators $\underline{1}$ and $\underline{2}$ have coordinates

$$\begin{bmatrix} 0 \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix}, \quad \begin{bmatrix} a_{12} \\ 0 \\ a_{32} \\ a_{42} \end{bmatrix},$$

respectively. Notice that three coordinates of v^{13} and $\underline{1}$ coincide and only the second one is different. Therefore, the tropical distance and Euclidean distance between these two points coincide, being $a_{41} - a_{42} - a_{21} = |a_{41} - a_{42} - a_{21}| = \operatorname{adiff}_A(24; 12)$, by inequalities (23). Moreover, we can check that the tropical distance between v^{13} and $\underline{2}$ is no less than $\operatorname{adiff}_A(24; 12)$, whence $v^{13} = \underline{12}$ and $v^{24} = \underline{21}$. Also, comparing v^{24} and $\underline{2}$, only the first coordinate is different. Therefore the tropical distance and Euclidean distance between these two points coincide, being $a_{32} - a_{31} - a_{12} = |a_{32} - a_{31} - a_{12}| = \operatorname{adiff}_A(13; 12)$. The tropical distance between points v^{13} and v^{24} is easily computed to be $\operatorname{adiff}_A(34; 12)$. Computations are similar if the type of line L_{12} is $[14, 23]$. This proves the second statement.

□

In the previous theorem, *the* 2×2 *tropical minors of* A *involving three or four different indices* come into play. There are 30 such minors in A. **Assume that all the** 2×2 **minors of** A **involving three or four different indices are tropically regular** (so that they have non–zero adiffs!). We call this **hypothesis 2**.

Lemma 6. *Assume hypotheses 1 and 2 for an order 4 matrix A. Then A is maximal and, moreover, all 2×2 minors of A are tropically regular.*

Proof. Suppose $\{i, j, k, l\} = [4]$. Then $L_{ij} \cap L_{ik} = \{\underline{i}\}$ and $L_{ij} \cap L_{kl} = \emptyset$, by hypothesis 1. Then by theorem 5, A is maximal.

Now, consider a minor involving only two indices, say $A(12, 12)$. It is tropically singular if and only if $a_{12} = a_{21} = 0$. If this is the case then, by lemma 4, L_{12} provides less that four extremals to $\operatorname{span}(A)$, so that A is not maximal. □

Remark 7. *Let four points in* $\mathbb{TP}^3$ *be given as the columns of a matrix* C*. The points are in* ***tropical general position*** *if, by definition, all the* $k \times k$ *minors of* C *are tropically regular, for all* $2 \leq k \leq 4$*.*

If A *satisfies hypotheses 1 and 2, then* A *is tropically regular, by [27], and all the* 2×2 *minors of* A *are tropically regular, by lemma 6. But, are all the* 3×3 *minors of* A *tropically regular, i.e., are the points represented by the columns of* A *in general position? We cannot answer this question yet.*

Example 8. *Here is a convex symmetric non–maximal example.*

$$A = \begin{bmatrix} 0 & -4 & -6 & -10 \\ -4 & 0 & -10 & -6 \\ -6 & -10 & 0 & -4 \\ -10 & -6 & -4 & 0 \end{bmatrix}. \tag{35}$$

The f–vector of $\operatorname{span}(A)$ *is* $(8, 14, 8)$*, the polygon–vector is* $(4, 4, 0, 0)$*. Lines* L_{13}*,* L_{14}*,* L_{23} *and* L_{24} *are of type* $[12, 34]$ *and* L_{12} *and* L_{34} *are of type* $[13, 24]$*; see figure 11.*

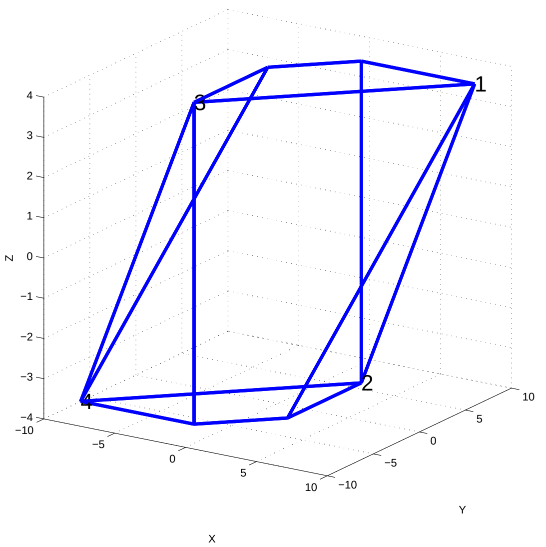

Figure 11: The tropical tetrahedron given by matrix (35).

3.7 Towards a classification of the combinatorial types of $\operatorname{span}(A)$

From now on, we will assume that A satisfies hypotheses 1 and 2.

Choose an oddly generated extremal in $\mathrm{span}(A)$ and look at the configuration at this point: we know that it is either $(5.5.5)$ right or left or $(p.q.r)$, with $\{p,q,r\} = \{4,5,6\}$. The following lemmas tell us that this configuration is encoded in the types of the 3 tropical lines passing through it.

Lemma 9. *Assume hypotheses 1 and 2. The following are equivalent:*

1. *the point* $\underline{123}$ *is* $(5.5.5)$ *right,*

2. L_{12} *is of type* $[14, 23]$*,* L_{23} *is of type* $[24, 13]$ *and* L_{31} *is of type* $[34, 12]$*.*

In such a case, the table below shows triples of points and the equations of the classical planes they classically span:

points	*equation*
$\underline{2}\ \underline{12}\ \underline{21}$	$X_3 - X_2 = a_{32}$
$\underline{3}\ \underline{23}\ \underline{32}$	$X_1 - X_3 = a_{13}$
$\underline{1}\ \underline{31}\ \underline{13}$	$X_2 - X_1 = a_{21}.$

Proof. From the right–hand–side of figure 4 we conclude that the direction of the bounded edge of line L_{12} is e_{23}, of line L_{23} is e_{31} and of line L_{31} is e_{12}. The equivalence follows from here.

Next, recall from elementary linear algebra that the equation of the plane generated by three points p, q, r in $\mathbb{R}^3$ is given by setting a certain 4×4 determinant, denoted $D(p, q, r)$, equal to zero: the first three rows of $D(p, q, r)$ are filled in with the coordinates of p, q, r and the coordinates of an indeterminate point $(X, X_2, X_3)^t$ and the fourth row is full of ones. The coordinates of $\underline{12}$ and $\underline{21}$ are easily obtained either from expressions (11) and (12) or from figure 4 and theorem 5. Now, we compute and factor the classical determinant $D(\underline{2}, \underline{12}, \underline{21})$, obtaining

$$\begin{aligned} &\begin{vmatrix} a_{12} - a_{42} & -a_{41} & -a_{41} & X_1 \\ -a_{42} & a_{31} - a_{41} - a_{32} & -a_{42} & X_2 \\ a_{32} - a_{42} & a_{31} - a_{41} & a_{32} - a_{42} & X_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} \\ = \quad & \pm\, \mathrm{adiff}_A(34;12)\, \mathrm{adiff}_A(14;24)(X_2 - X_3 + a_{32}). \end{aligned}$$

By hypothesis 2, the adiffs in the line right above are non–zero, so that the equation reduces to $X_3 - X_2 = a_{32}$. The computations are similar for other classical planes. ∎

Lemma 10. *Assume hypotheses 1 and 2 for* A*. The following are equivalent:*

1. *the point* $\underline{123}$ *is* $(5.5.5)$ *left,*

2. L_{12} *is of type* $[13, 24]$, L_{23} *is of type* $[12, 34]$ *and* L_{31} *is of type* $[14, 23]$.

In such a case, the table below shows triples of points and the equations of the classical planes they classically span:

points	*equation*
$\underline{1}$ $\underline{12}$ $\underline{21}$	$X_3 - X_1 = a_{31}$
$\underline{2}$ $\underline{23}$ $\underline{32}$	$X_1 - X_2 = a_{12}$
$\underline{3}$ $\underline{31}$ $\underline{13}$	$X_2 - X_3 = a_{23}$. $\square$

$\square$

Lemma 11. *Assume hypotheses 1 and 2 for* A. *The following are equivalent:*

1. *the point* $\underline{4}$ *is* $(5.5.5)$ *right,*

2. L_{14} *is of type* $[12, 34]$, L_{24} *is of type* $[14, 23]$ *and* L_{34} *is of type* $[13, 24]$.

In such a case, the table below shows triples of points and the equations of the classical planes they classically span:

points	*equation*
$\underline{1}$ $\underline{14}$ $\underline{41}$	$X_2 - X_1 = a_{21}$
$\underline{2}$ $\underline{24}$ $\underline{42}$	$X_3 - X_2 = a_{32}$
$\underline{3}$ $\underline{34}$ $\underline{43}$	$X_1 - X_3 = a_{13}$. $\square$

Lemma 12. *Assume hypotheses 1 and 2 for* A. *The following are equivalent:*

1. *the point* $\underline{4}$ *is* $(5.5.5)$ *left,*

2. L_{14} *is of type* $[13, 24]$, L_{24} *is of type* $[12, 34]$ *and* L_{34} *is of type* $[14, 23]$.

In such a case, the table below shows triples of points and the equations of the classical planes they classically span:

points	*equation*
$\underline{1}$ $\underline{14}$ $\underline{41}$	$X_3 - X_1 = a_{31}$
$\underline{2}$ $\underline{24}$ $\underline{42}$	$X_1 - X_2 = a_{12}$
$\underline{3}$ $\underline{34}$ $\underline{43}$	$X_2 - X_3 = a_{23}$. $\square$

Similarly, looking at figure 7, we can prove

- $\underline{123}$ is $(4.6.5)$ if and only if L_{12} is $[13, 24]$, L_{23} is $[13, 24]$ and L_{31} is $[12, 34]$,
- $\underline{123}$ is $(6.4.5)$ if and only if L_{12} is $[23, 14]$, L_{23} is $[12, 34]$ and L_{31} is $[23, 14]$.

Analogous statements can be proved for $\underline{123}$ and $\underline{4}$ with any configuration $(p.q.r)$, with $\{p, q, r\} = \{4, 5, 6\}$.

By the lemmas and comment above, the types of L_{12}, L_{23}, L_{31} (resp. L_{14}, L_{24}, L_{34}) determine the configuration at point $\underline{123}$ (resp. $\underline{4}$) and conversely. This leads us to define the **type–vector** $t = (t_1, t_2, t_3)$, where t_j is the number of tropical lines of type $[4j, kl]$, with $\{j, k, l\} = \{1, 2, 3\}$. By theorem 1, both for $\underline{123}$ and $\underline{4}$, either the three types are all different or just two of them are equal. Thus, up to a permutation, t equals one of the following

$$(2, 2, 2), \quad (3, 2, 1), \quad (3, 3, 0), \quad (4, 1, 1), \quad (4, 2, 0). \tag{36}$$

Hexagons in $\operatorname{span}(A)$**.** It is easy to realize that two lines of the same type (say $[ik, jl]$) having concatenated indices (say L_{ij}, L_{jk}) yield one hexagon in $\operatorname{span}(A)$ at extremal $\underline{ijk}$. However, two lines of the same type having disjoint indices (say L_{ij}, L_{kl}, with $\{i, j, k, l\} = [4]$) yield no hexagon at all. And, what happens if two or more hexagons are adjacent facets of $\operatorname{span}(A)$? Fix a type, e.g. $[ik, jl]$.

- Three lines of the same type having concatenated indices (say L_{ij}, L_{jk}, L_{kl}) yield two adjacent hexagons. The converse is true.
- Four lines of the same type necessarily have concatenated indices (say L_{ij}, L_{jk}, L_{kl}, L_{li}) and yield four adjacent hexagons closing up into a cycle. The converse is true.

3.8 Searching for $\operatorname{span}(A)$ with the combinatorial type of a regular dodecahedron

We seek an order 4 Kleene star matrix A having f–vector $(20, 30, 12)$ and polygon–vector $(0, 0, 12, 0)$. We will not find any!

By a translation and a change of coordinates, we can assume that the coordinates of $\underline{123}$ and $\underline{4}$ are $[0, 0, 0, 0]^t$ and $[-a, -b, -c, 0]^t$, respectively, with $0 < a \leq b \leq c$. If $\operatorname{span}(A)$ must have 12 pentagonal facets, then we can assume that points $\underline{123}$ and $\underline{4}$ are both $(5.5.5)$; this way $\operatorname{span}(A)$ has, at least, 6 pentagonal facets. Four cases arise, depending on whether the points $\underline{123}$ and $\underline{4}$ are left or right. These are dealt with in theorems 13, 14 and 15.

Theorem 13. *If $\underline{123}$ and $\underline{4}$ are both $(5.5.5)$ left, then polygon–vector of* $\operatorname{span}(A)$ *is* $(0, 3, 6, 3)$.

Proof. By lemmas 10 and 12, the type–vector is $(2, 2, 2)$ and

- points $\underline{1}$, $\underline{14}$, $\underline{41}$, $\underline{12}$, $\underline{21}$ lie on the classical plane $X_3 - X_1 = a_{31}$,
- points $\underline{2}$, $\underline{24}$, $\underline{42}$, $\underline{23}$, $\underline{32}$ lie on the classical plane $X_1 - X_2 = a_{12}$,

- points $\underline{3}$, $\underline{34}$, $\underline{43}$, $\underline{31}$, $\underline{13}$ lie on the classical plane $X_2 - X_3 = a_{23}$.

Moreover, the coordinates of $\underline{134} = \mathrm{col}((-A^t)_0, 2)$ also satisfy the equation $X_2 - X_3 = a_{23}$, so that $\underline{3}$, $\underline{34}$, $\underline{43}$, $\underline{31}$, $\underline{13}$ and $\underline{134}$ make up a hexagon. Similar for the points $\underline{124}$, $\underline{234}$, and so $\mathrm{span}(A)$ has three hexagons, three pentagons and three quadrangles. □

Theorem 14. *If $\underline{123}$ and $\underline{4}$ are both $(5.5.5)$ right, then polygon–vector of $\mathrm{span}(A)$ is $(0, 3, 6, 3)$.* □

Theorem 15. *It is not possible to have A satisfying hypotheses 1 and 2 such that $\underline{123}$ is $(5.5.5)$ left and $\underline{4}$ is $(5.5.5)$ right or $\underline{123}$ is $(5.5.5)$ right and $\underline{4}$ is $(5.5.5)$ left.*

Proof. By symmetry, it is enough to address the case where $\underline{123}$ is $(5.5.5)$ left and $\underline{4}$ is $(5.5.5)$ right. If $\underline{123}$ is $(5.5.5)$ left then, using lemma 10, we have

$$\underline{12} = \begin{bmatrix} -a_{41} \\ -a_{42} \\ a_{31} - a_{41} \\ 0 \end{bmatrix}, \underline{23} = \begin{bmatrix} a_{12} - a_{42} \\ -a_{42} \\ -a_{43} \\ 0 \end{bmatrix}, \underline{31} = \begin{bmatrix} -a_{41} \\ a_{23} - a_{43} \\ -a_{43} \\ 0 \end{bmatrix},$$

$$\underline{21} = \begin{bmatrix} a_{32} - a_{42} - a_{31} \\ -a_{42} \\ a_{32} - a_{42} \\ 0 \end{bmatrix}, \underline{32} = \begin{bmatrix} a_{13} - a_{43} \\ a_{13} - a_{43} - a_{12} \\ -a_{43} \\ 0 \end{bmatrix}, \underline{13} = \begin{bmatrix} -a_{41} \\ a_{21} - a_{41} \\ a_{21} - a_{41} - a_{23} \\ 0 \end{bmatrix}.$$

Since $\underline{4}$ is $(5.5.5)$ right then, using lemma 11, we have

$$\underline{14} = \begin{bmatrix} a_{34} - a_{31} \\ a_{34} - a_{31} + a_{21} \\ a_{34} \\ 0 \end{bmatrix}, \underline{24} = \begin{bmatrix} a_{14} \\ a_{14} - a_{12} \\ a_{14} - a_{12} + a_{32} \\ 0 \end{bmatrix}, \underline{34} = \begin{bmatrix} a_{24} - a_{23} + a_{13} \\ a_{24} \\ a_{24} - a_{23} \\ 0 \end{bmatrix},$$

$$\underline{41} = \begin{bmatrix} a_{24} - a_{21} \\ a_{24} \\ a_{34} \\ 0 \end{bmatrix}, \underline{42} = \begin{bmatrix} a_{14} \\ a_{34} - a_{32} \\ a_{34} \\ 0 \end{bmatrix}, \underline{43} = \begin{bmatrix} a_{14} \\ a_{24} \\ a_{14} - a_{13} \\ 0 \end{bmatrix}.$$

We can assume that the coordinates of $\underline{123}$ are $[0, 0, 0, 0]^t$ and the coordinates of $\underline{4}$ are $[-a, -b, -c, 0]^t$, for some positive a, b, c. Then, by expression (25),

$$A = A_0 = \begin{bmatrix} 0 & -\gamma_2 - \delta_2 & -\gamma_3 - \gamma_2 - \delta_2 & -a \\ -\gamma_1 - \gamma_3 - \delta_3 & 0 & -\gamma_3 - \delta_3 & -b \\ -\gamma_1 - \delta_1 & -\gamma_2 - \gamma_1 - \delta_1 & 0 & -c \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (37)$$

Substituting a_{ij} by its value in the coordinates of $\underline{14}$, we get

$$\underline{14} = \begin{bmatrix} -c+\gamma_1+\delta_1 \\ -c+\delta_1-\gamma_3-\delta_3 \\ -c \\ 0 \end{bmatrix},$$

so that the vector $\overrightarrow{\underline{14}\,\underline{1}}$ equals $(c-\gamma_1-\delta_1)e_{123}$. Similarly, we see that the vectors $\overrightarrow{\underline{24}\,\underline{2}}$ and $\overrightarrow{\underline{34}\,\underline{3}}$ have direction e_{123}.

We use lemma 11 and theorem 5 to obtain

- $\mathrm{tdist}(\underline{14},\underline{41}) = \mathrm{adiff}_A(23,14) = |-\gamma_3-\delta_3+\delta_1-c+b| > 0$,
- $\mathrm{dist}(\underline{1},\underline{14}) = \mathrm{adiff}_A(34,14) = |\gamma_1+\delta_3-c| > 0$,
- $\mathrm{dist}(\underline{4},\underline{41}) = \mathrm{adiff}_A(12,14) = |\gamma_1+\gamma_3+\delta_3+a-b| > 0$,
- $\mathrm{tdist}(\underline{24},\underline{42}) = \mathrm{adiff}_A(13,24) = |-\gamma_1-\delta_1+\delta_2-a+c| > 0$,
- $\mathrm{dist}(\underline{2},\underline{24}) = \mathrm{adiff}_A(14,24) = |\gamma_2+\delta_2-a| > 0$,
- $\mathrm{dist}(\underline{4},\underline{42}) = \mathrm{adiff}_A(23,24) = |\gamma_1+\delta_1+\gamma_2+b-c| > 0$,
- $\mathrm{tdist}(\underline{34},\underline{43}) = \mathrm{adiff}_A(12,34) = |-\gamma_2-\delta_2+\delta_3-b+a| > 0$,
- $\mathrm{dist}(\underline{3},\underline{34}) = \mathrm{adiff}_A(24,34) = |\gamma_3+\delta_3-b| > 0$,
- $\mathrm{dist}(\underline{4},\underline{43}) = \mathrm{adiff}_A(13,34) = |\gamma_2+\delta_2+\gamma_3+c-a| > 0$.

For each $j=1,2,3$, it is obvious that

$$\underline{4} + \overrightarrow{\underline{4}\,\underline{4j}} + \overrightarrow{\underline{4j}\,\underline{j4}} + \overrightarrow{\underline{j4}\,\underline{j}} = \underline{j}; \tag{38}$$

see figure 12, for $j=1$.

Having in mind tropical distances and the directions of the vectors e_i, e_{ij}, e_{123}, equalities (38) convert into the following ones

$$\begin{bmatrix} -a+|\gamma_1+\gamma_3+\delta_3+a-b|+|-\gamma_3-\delta_3+\delta_1-c+b|+|\gamma_1+\delta_1-c| \\ -b+|-\gamma_3-\delta_3+\delta_1-c+b|+|\gamma_1+\delta_1-c| \\ -c+|\gamma_1+\delta_1-c| \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\gamma_1-\gamma_3-\delta_3 \\ -\gamma_1-\delta_1 \\ 0 \end{bmatrix} \tag{39}$$

$$\begin{bmatrix} -a+|\gamma_2+\delta_1-a| \\ -b+|\gamma_1+\delta_1+\gamma_2+b-c|+|-\gamma_1-\delta_1+\delta_2-a+c|+|\gamma_2+\delta_1-a| \\ -c+|-\gamma_1-\delta_1+\delta_2-a+c|+|\gamma_2+\delta_1-a| \\ 0 \end{bmatrix} = \begin{bmatrix} -\gamma_2-\delta_2 \\ 0 \\ -\gamma_2-\gamma_1-\delta_1 \\ 0 \end{bmatrix} \tag{40}$$

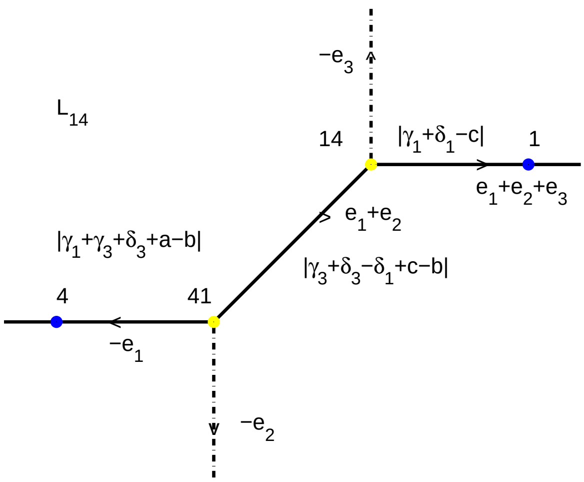

Figure 12: Going from point 4 to point 1 along the tropical line L_{14}.

$$\begin{bmatrix} -a+|\gamma_2+\delta_2-\delta_3+b-a|+|\gamma_3+\delta_3-b| \\ -b+|\gamma_3+\delta_3-b| \\ -c+|\gamma_2+\delta_2+\gamma_3+c-a|+|\gamma_2+\delta_2-\delta_3+b-a|+|\gamma_3+\delta_3-b| \\ 0 \end{bmatrix} = \begin{bmatrix} -\gamma_3-\gamma_2-\delta_2 \\ -\gamma_3-\delta_3 \\ 0 \\ 0 \end{bmatrix} \tag{41}$$

Working through equalities (39),(40) and (41), and using that no absolute value vanishes (due to maximality of A), it follows that

$$\begin{aligned} \gamma_1+\delta_1 &< c, \\ \gamma_2+\delta_2 &< a, \\ \gamma_3+\delta_3 &< b, \\ \gamma_1+\delta_1-\delta_2 &< c-a, \\ \gamma_2+\delta_2-\delta_3 &< a-b, \\ \gamma_3+\delta_3-\delta_1 &< b-c, \end{aligned}$$

but then $\gamma_1+\gamma_2+\gamma_3<0$, contradicting $\gamma_j>0$, for all $j=1,2,3$.

□

3.9 Classification

For $\mathrm{span}(A)$ convex and maximal, here is a list of **additional properties** it enjoys (with a short explanation provided):

1. each facet of $\mathrm{span}(A)$ contains exactly one generator, exactly one 3–generated extremal and 1, 2 or 3 2–generated extremals, (this is true for facets meeting $\underline{123}$ in $\mathrm{span}(A)$, thus it is true for facets meeting $\underline{4}$ in $\mathrm{span}(A^t)$, thus it is true for facets meeting any oddly generated extremal, which are all of the facets),

2. at an oddly generated extremal no two hexagons meet, (same reasons as in item above),

3. $f_4 = f_6$, i.e., the number of quadrangles in $\mathrm{span}(A)$ equals the number of hexagons, (this is true because the f–vector of $\mathrm{span}(A)$ is $(20, 30, 12)$, the same f–vector the regular dodecahedron $\mathcal{D}$ has. Since $\mathrm{span}(A)$ is trivalent (by [18]), then the combinatorial type of $\mathrm{span}(A)$ can be obtained from $\mathcal{D}$ by a finite sequence of combinatorial polyhedral transformations; see [12]).

Corollary 16. *It is not possible to have* A *with f–vector* $(20, 30, 12)$ *and polygon–vector either* $(0, 0, 12, 0)$ *or* $(0, 1, 10, 1)$. *In particular,* $\mathrm{span}(A)$ *does not have the combinatorial type of the regular dodecahedron* $\mathcal{D}$*, for any* A.

Proof. Assume that A satisfies hypotheses 1 and 2, has f–vector $(20, 30, 12)$, $f_4 = f_6$ and $f_5 \geq 10$. There are three cases:

- all the facets of $\mathrm{span}(A)$ are pentagons,i.e, $f_5 = 12$,
- a quadrangle and a hexagon are adjacent facets in $\mathrm{span}(A)$,
- a quadrangle and a hexagon are non–adjacent facets in $\mathrm{span}(A)$.

In the third case, $6 + 4 = 10$ extremals of $\mathrm{span}(A)$ meet either a quadrangle or a hexagon and $f_5 \geq 10$, so that the remaining 10 extremals in $\mathrm{span}(A)$ must have configuration $(5.5.5)$. Then, there must exist two opposite extremals (recall definition in p. 18) of $\mathrm{span}(A)$ both of which are $(5.5.5)$. But then, theorems 13, 14 and 15 tell us that this cannot happen. In the first two cases, the existence of two opposite extremals having configuration $(5.5.5)$ is even more obvious. ☐

By corollary 16 and item 1, we have $2 \leq f_6 \leq 4$. Thus, the polygon–vector $(0, f_4, f_5, f_6)$ of $\mathrm{span}(A)$ is

$$(0, 2, 8, 2), \qquad (0, 3, 6, 3), \qquad (0, 4, 4, 4). \tag{42}$$

Our classification goes according to the type–vector (see p. 25) and number and adjacency of hexagons (the polygon–vector is not enough to classify!). Up to symmetry and changes of coordinates, the combinatorial type of $\mathrm{span}(A)$ is classified as follows:

Class 1. If $t = (2,2,2)$, then by theorems 5 and 15, the points $\underline{123}$ and $\underline{4}$ are both $(5.5.5)$ left or both $(5.5.5)$ right. In any case, the polygon–vector is $(0,3,6,3)$, by theorems 13 and 14. No pairs of adjacent hexagons exist in $\operatorname{span}(A)$, since the indices of lines corresponding to any given type are disjoint. For examples, see lemma 17.

Class 2. If $t = (3,2,1)$ and the indices of the lines of type $[13,24]$ are concatenated, then $\operatorname{span}(A)$ has $2+1=3$ hexagons, so that the polygon–vector is $(0,3,6,3)$. Two hexagons are adjacent. See example 21.

Class 3. If $t = (3,2,1)$ and the indices of the lines of type $[13,24]$ are disjoint, then $\operatorname{span}(A)$ has 2 hexagons, which are adjacent. The polygon–vector is $(0,2,8,2)$. See example 20.

Class 4. If $t = (3,3,0)$, then $\operatorname{span}(A)$ has 2 pairs of adjacent hexagons. The polygon–vector is $(0,4,4,4)$. For examples, see lemma 18.

Class 5. If $t = (4,1,1)$, then $\operatorname{span}(A)$ has a cycle of 4 adjacent hexagons. The polygon–vector is $(0,4,4,4)$. For examples, see lemma 18. In this case, the configurations at $\underline{123}$ and $\underline{4}$ are not equal: one is $(p.q.r)$ and the other is $(r.q.p)$, for some $\{p,q,r\} = \{4,5,6\}$.

Class 6. If $t = (4,2,0)$, then $\operatorname{span}(A)$ has a cycle of 4 adjacent hexagons. The indices of the two lines of type $[13,24]$ are disjoint. The polygon–vector is $(0,4,4,4)$. In this case, the configurations at $\underline{123}$ and $\underline{4}$ are equal to $(p.q.r)$, for some $\{p,q,r\} = \{4,5,6\}$. For examples, see lemma 18.

Symmetry and chirality. The symmetric image of configuration $(5.5.5)$ left is $(5.5.5)$ right. The symmetric image of configuration $(p.q.r)$ is $(p.q.r)$. Thus, $\operatorname{span}(A)$ admits a central symmetry only for class 6.

If in $\operatorname{span}(A)$, the points $\underline{123}$ and $\underline{4}$ are both $(5.5.5)$ right, then in $\operatorname{span}(A^t)$, the points $\underline{123}$ and $\underline{4}$ are both $(5.5.5)$ left and, thus, the solids $\operatorname{span}(A), \operatorname{span}(A^t)$ are chiral to each other. This happens in class 1.

3.10 Compatible configurations at $\underline{123}$ and at $\underline{4}$ and examples

Fix an order 4 Kleene star A. As we saw in the proof of theorem 1, the type of the line L_{ij} depends on the value attained by the tropical minor $M(ij) := |A(kl,ij)|_{\text{trop}}$, where $\{i,j,k,l\} = [4]$. More explicitly,

$$M(12) = \max\{a_{31}+a_{42}, a_{41}+a_{32}\} = |A(34,12)|_{\text{trop}}, \tag{43}$$

$$M(13) = \max\{a_{21}+a_{43}, a_{41}+a_{23}\} = |A(24,13)|_{\text{trop}}, \tag{44}$$

$$M(14) = \max\{a_{21}+a_{34}, a_{31}+a_{24}\} = |A(23,14)|_{\text{trop}}, \tag{45}$$

$$M(23) = \max\{a_{12} + a_{43}, a_{42} + a_{13}\} = |A(14, 23)|_{\mathrm{trop}}, \tag{46}$$
$$M(24) = \max\{a_{12} + a_{34}, a_{32} + a_{14}\} = |A(13, 24)|_{\mathrm{trop}}, \tag{47}$$
$$M(34) = \max\{a_{13} + a_{24}, a_{23} + a_{14}\} = |A(12, 34)|_{\mathrm{trop}}. \tag{48}$$

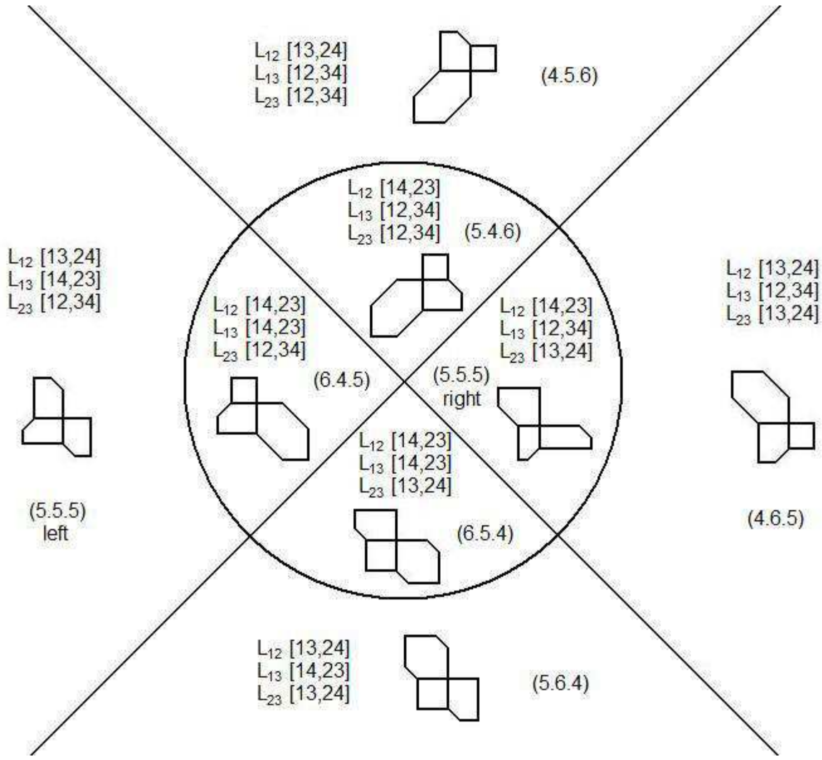

Figure 13: Possible configurations at point $\underline{123}$. The circle and the two straight lines represent the classical hyperplanes $a_{31} + a_{42} = a_{41} + a_{32}$, $a_{21} + a_{43} = a_{41} + a_{23}$ and $a_{12} + a_{43} = a_{42} + a_{13}$.

By inequalities (23), the order 4 Kleene stars form a closed convex subset $\overline{\mathcal{S}}$ of $\mathbb{R}^{12}_{\leq 0}$. Those matrices satisfying hypotheses 1 and 2 form an open dense subset $\mathcal{S}$ in $\overline{\mathcal{S}}$. For any $A \in \mathcal{S}$, the combinatorial type of $\mathrm{span}(A)$ changes whenever the value of a maximum among (43)–(48) changes. Therefore, the following family of

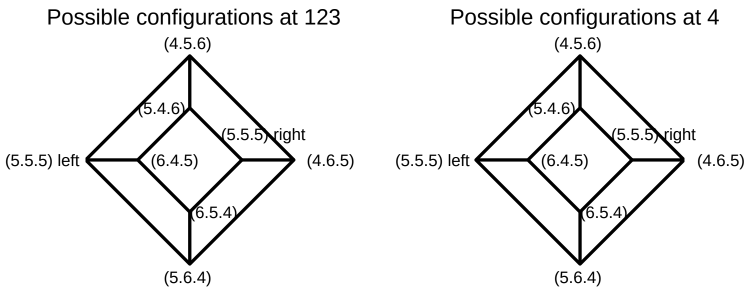

Figure 14: Dual graphs of cell decompositions at $\underline{123}$ and at $\underline{4}$.

hyperplanes splits $\mathcal{S}$ into open cells:

$$\begin{aligned} a_{31} + a_{42} &= a_{41} + a_{32} \\ a_{21} + a_{43} &= a_{41} + a_{23} \\ a_{21} + a_{34} &= a_{31} + a_{24} \\ a_{12} + a_{43} &= a_{42} + a_{13} \\ a_{12} + a_{34} &= a_{32} + a_{14} \\ a_{13} + a_{24} &= a_{23} + a_{14}. \end{aligned}$$

The situation around point $\underline{123}$ is depicted in figure 13. The situation around point $\underline{4}$ is similar. The dual graphs of both cell decompositions appear in figure 14; each node of the graph on the left (resp. right) corresponds to a possible configuration at $\underline{123}$ (resp. at $\underline{4}$).

We say that given configurations at $\underline{123}$ and at $\underline{4}$ are **compatible** if there exists $A \in \mathcal{S}$ such that $\operatorname{span}(A)$ realizes both. To obtain examples of some combinatorial types of $\operatorname{span}(A)$, it is enough to find certain compatible pairs of configurations at $\underline{123}$ and at $\underline{4}$. Theorems 13 and 14 show two compatible configurations (symmetric one to another), while theorem 15 shows some incompatible configurations.

For each class we can always find examples with integer matrices (due to translations and scaling). Then, the tropical distance between neighboring extremals is the integer length of the edge they span.

Revisiting theorems 13 and 14, we find matrices $A \in \mathcal{S}$ with type–vector

$(2,2,2)$, thus in class 1. The following provides simple examples: e.g., take $\gamma = \delta = 1$, $c = 2$.

Lemma 17. *Suppose that* γ, δ, c *are positive reals such that* $2\gamma + \delta < 2c$. *Then*

$$A = \begin{bmatrix} 0 & -2\gamma - \delta & -\gamma - \delta & -c \\ -\gamma - \delta & 0 & -2\gamma - \delta & -c \\ -2\gamma - \delta & -\gamma - \delta & 0 & -c \\ -c & -c & -c & 0 \end{bmatrix} \tag{49}$$

$\operatorname{span}(A)$ *belongs to class 1.*

Proof. The matrix A is normal and satisfies inequalities (23), so that it is tropically idempotent. By expression (26), the point $\underline{123}$ is $(5.5.5)$ right, with $\gamma_j = \gamma$, $\delta_j = \delta$, for $j = 1, 2, 3$.

Direct computations yield that lines L_{13} and L_{14} are of type $[12, 34]$, lines L_{23} and L_{34} are of type $[13, 24]$ and lines L_{12} and L_{24} are of type $[14, 23]$. This implies that the point $\underline{4}$ is $(5.5.5)$ right, and then the result follows from theorem 14. ☐

We can make additional computations for matrix (49). Lemma 2 provides the coordinates of the points $\underline{ij4}$. We compute the coordinates of the points below, obtaining

$$\underline{14} = \begin{bmatrix} 2\gamma + \delta - c \\ \gamma - c \\ -c \\ 0 \end{bmatrix}, \quad \underline{41} = \begin{bmatrix} \gamma + \delta - c \\ -c \\ -c \\ 0 \end{bmatrix}, \tag{50}$$

$$\underline{24} = \begin{bmatrix} -c \\ 2\gamma + \delta - c \\ \gamma - c \\ 0 \end{bmatrix}, \quad \underline{42} = \begin{bmatrix} -c \\ \gamma + \delta - c \\ -c \\ 0 \end{bmatrix}, \tag{51}$$

$$\underline{34} = \begin{bmatrix} \gamma - c \\ -c \\ 2\gamma + \delta - c \\ 0 \end{bmatrix}, \quad \underline{43} = \begin{bmatrix} -c \\ -c \\ \gamma + \delta - c \\ 0 \end{bmatrix}, \tag{52}$$

showing that $\underline{4}$, $\underline{234}$, $\underline{24}$, $\underline{42}$, $\underline{43}$ lie on $X_1 = -c$, $\underline{4}$, $\underline{134}$, $\underline{34}$, $\underline{43}$, $\underline{41}$ lie on $X_2 = -c$ and $\underline{4}$, $\underline{124}$, $\underline{14}$, $\underline{41}$, $\underline{42}$ lie on $X_3 = -c$.

Given $p = [p_1, p_2, p_3, p_4]^t$, we consider the corresponding **circulant** and **anti-circulant symmetric** matrices

$$C(p) = \begin{bmatrix} p_1 & p_4 & p_3 & p_2 \\ p_2 & p_1 & p_4 & p_3 \\ p_3 & p_2 & p_1 & p_4 \\ p_4 & p_3 & p_2 & p_1 \end{bmatrix}, \qquad A(p) = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_2 & p_1 & p_4 & p_3 \\ p_3 & p_4 & p_1 & p_2 \\ p_4 & p_3 & p_2 & p_1 \end{bmatrix}. \tag{53}$$

In the lemma below, take for instance, $a = 3, c = 6$ and $b = 4$ or $b = 5$.

Lemma 18. *Consider numbers* a, b, c *such that* $0 < a < b < c \leq 2a$ *and* $2b \neq a+c$ *and set* $p = [0, -a, -b, -c]^t$. *Then*

1. $\operatorname{span} C(p)$ *belongs to classes 4 or 5,*
2. $\operatorname{span} A(p)$ *belongs to the class 6.*

Proof. The proofs of both items are similar. We will only prove item 1. The hypothesis on a, b, c guarantee that $C(p)$ is a Kleene star and all the 2×2 minors of $C(p)$ are tropically regular. Thus, $C(p)$ satisfies hypothesis 2. Concerning hypothesis 1, notice that the vertex of $\Pi := L(\underline{1}, \underline{2}, \underline{3})$ is $\operatorname{col}(-C(p)^t, 4) = [c, b, a, 0]^t$, so that the tropical linear form corresponding to Π is

$$-c \odot X_1 \oplus -b \odot X_2 \oplus -a \odot X_3 \oplus 0 \odot X_4 = \max\{X_1 - c, X_2 - b, X_3 - a, X_4\}$$

and, plugging in the coordinates of $\underline{4}$, we get that the maximum

$$\max\{-a - c, -2b, -a - c, 0\} = 0$$

is attained only once, showing that the columns of $C(p)$ are not coplanar, tropically.

The type of L_{13} is $[12, 34]$ and the type of L_{24} is $[14, 23]$. The remaining computations depend on

$$M = \max\{-2b, -a - c\},$$

the value of any of the 2×2 tropical minors $A(ij, kl)$ with four different indices.

If $M = -2b$, then the type of lines L_{12}, L_{14} L_{23} and L_{34} is $[13, 24]$. Thus, the type–vector is $(1, 4, 1)$. Otherwise, $M = -a - c$ and the type of lines L_{12} and L_{34} is $[14, 23]$, while the type of lines L_{14} and L_{23} is $[12, 34]$. The type–vector is $(3, 0, 3)$, now. In any case, the polygon–vector is $(0, 4, 4, 4)$. $\square$

Here are more computations for $\operatorname{span} C(p)$: the coordinates of the points $\underline{ij}$ and the tropical distances between points, obtaining:

- $\operatorname{tdist}(\underline{13}, \underline{31}) = \operatorname{tdist}(\underline{24}, \underline{42}) = 2(c - a)$,
- $\operatorname{dist}(\underline{1}, \underline{13}) = \operatorname{dist}(\underline{31}, \underline{3}) = \operatorname{dist}(\underline{2}, \underline{24}) = \operatorname{dist}(\underline{42}, \underline{4}) = a + b - c$,
- $\operatorname{tdist}(\underline{ij}, \underline{ji}) = |a - 2b + c|$, for other choices of i, j,
- $\operatorname{tdist}(\underline{i}, \underline{ij})$ are: $b + c - a$, $c + a - b$, $2c - b$, $2c - a$, $2b - a$, $2b - c$, $2a - b$ and $2a - c$, for other choices of i, j.

These distances are all strictly positive, by maximality. Moreover,

$$\underline{13} = \begin{bmatrix} c \\ c-a \\ a \\ 0 \end{bmatrix}, \underline{31} = \begin{bmatrix} 2a-c \\ a-c \\ a \\ 0 \end{bmatrix}, \tag{54}$$

$$\underline{24} = \begin{bmatrix} -a \\ c-a \\ c-2a \\ 0 \end{bmatrix}, \underline{42} = \begin{bmatrix} -a \\ a-c \\ -c \\ 0 \end{bmatrix}, \tag{55}$$

Consider the point $\underline{123}$, of coordinates $[c, b, a, 0]^t$, and all the nine points around it: $\underline{1}$, $\underline{2}$, $\underline{3}$, $\underline{12}$, $\underline{21}$, $\underline{13}$, $\underline{31}$, $\underline{23}$ and $\underline{32}$.

Assume that $M = -2b$. Computing coordinates, we get

$$\underline{12} = \begin{bmatrix} c \\ b \\ c-b \\ 0 \end{bmatrix}, \underline{21} = \begin{bmatrix} 2b-a \\ b \\ b-a \\ 0 \end{bmatrix},$$

$$\underline{23} = \begin{bmatrix} b-c \\ b \\ 2b-c \\ 0 \end{bmatrix}, \underline{32} = \begin{bmatrix} a-b \\ b \\ a \\ 0 \end{bmatrix}.$$

Thus

- points $\underline{123}$, $\underline{1}$, $\underline{12}$ and $\underline{13}$ lie in the classical plane of equation $X_1 = c$, making a quadrangle,
- points $\underline{123}$, $\underline{31}$, $\underline{13}$, $\underline{3}$ and $\underline{32}$ lie in the classical plane of equation $X_3 = a$, making a pentagon,
- points $\underline{123}$, $\underline{12}$, $\underline{21}$, $\underline{2}$, $\underline{23}$, and $\underline{32}$ lie in the classical plane of equation $X_2 = b$, making a hexagon.

If $M = -a - c$, then

$$\underline{12} = \begin{bmatrix} c \\ a-b+c \\ c-b \\ 0 \end{bmatrix}, \underline{21} = \begin{bmatrix} c \\ b \\ b-a \\ 0 \end{bmatrix},$$

$$\underline{23} = \begin{bmatrix} b-c \\ b \\ a \\ 0 \end{bmatrix}, \underline{32} = \begin{bmatrix} a-b \\ a-b+c \\ a \\ 0 \end{bmatrix}.$$

Thus

- points 123, 1, 12, 21 and 13 make a pentagon in $X_1 = c$,
- points 123, 23, 2 and 21 make a quadrangle in $X_2 = b$,
- points 123, 23, 32, 3, 31, and 13 make a hexagon in $X_3 = a$.

Around the points 124, 134, 234 the situation is similar to 123, by a change of coordinates.

Notice that $\operatorname{span} C(p)$ is not maximal, if $2b = a + c$.

Remark 19. *If $M = -2b$, then the vertices of the lines L_{ij} can be arranged into two more circulant matrices of vectors $[0, c, b, c-b]^t$ and $[0, a-b, b, a]^t$, with corresponding columns 23, 34, 41, 12, and 43, 14, 21, 32.*

Similarly, for $M = -a-c$, $[0, b-c, b, a]^t$ and $[0, c, b, b-a]^t$, 34, 41, 12, 23, and 32, 43, 14, 21.

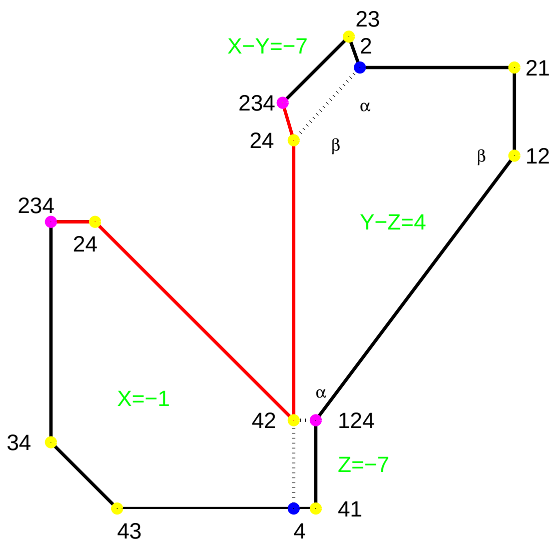

Figure 15: Hexagonal adjacent facets of example 20. Dotted lines must be valley–folded. Red segments must be glued together. Angles $\alpha \simeq 125°16'$ and $\beta \simeq 144°44'$ are defined in p.15. Other angles in this picture are $45°, 90°$ or $135°$ and $\overline{\alpha}$.

Example 20. *For*

$$A = \begin{bmatrix} 0 & -7 & -5 & -1 \\ -8 & 0 & -8 & -7 \\ -7 & -4 & 0 & -7 \\ -9 & -9 & -8 & 0 \end{bmatrix}, \qquad A' = \begin{bmatrix} 0 & -6 & -6 & -4 \\ -4 & 0 & -9 & -6 \\ -6 & -8 & 0 & -7 \\ -10 & -7 & -6 & 0 \end{bmatrix} \tag{56}$$

$\mathrm{span}(A)$, $\mathrm{span}(A')$ *belong both to class 3. This class is not found in [18].*

Let us check the details for A. *The tropical lines* L_{12}, L_{24}, L_{34} *are of type* $[14, 23]$, L_{23}, L_{14} *are of type* $[13, 24]$ *and* L_{13} *is of type* $[12, 34]$. *Thus, the type–vector is* $t = (1, 2, 3)$. *The indices of the lines of type* $[13, 24]$ *are disjoint (*23 *and* 14*). Therefore,* $\mathrm{span}(A)$ *contains only two hexagonal facets; see figure 15. These are* $\underline{2}$,$\underline{12}$, $\underline{21}$, $\underline{24}$, $\underline{42}$, $\underline{124}$ *on* $X_2 - X_3 = 4$ *and* $\underline{4}$, $\underline{24}$, $\underline{34}$,$\underline{42}$, $\underline{43}$, $\underline{234}$ *on* $X_1 = -1$. *The common segment joins* $\underline{24}$ *to* $\underline{42}$.

The two quadrangular facets are $\underline{2}$, $\underline{23}$, $\underline{24}$, $\underline{234}$ *on the plane on* $X_1 - X_2 = -7$, *and* $\underline{4}$, $\underline{41}$, $\underline{42}$, $\underline{124}$ *on* $X_3 = -7$.

Notice that at $\underline{24}$, *two hexagons and a quadrilateral meet, and so a total of 12 extremals appear in the configuration of* $\underline{24}$. *However, we know that, whatever the configuration at any oddly generated extremal is, exactly ten extremals appear in it; see p. 22. Therefore, this solid is* ***not vertex–transitive***.

The angle–vector (see p.15) of the hexagon contained in $X_2 - X_3 = 4$ *is* $\langle 90, \alpha, \beta, 90, \alpha, \beta \rangle$, *with* $\alpha \simeq 125°16'$ *and* $\beta \simeq 35°16'$, *and* $2(90 + \alpha + \beta) = 720 = 180 \times (6 - 2)$ *is the sum of the interior angles of a hexagon.*

Example 21.

$$A = \begin{bmatrix} 0 & -6 & -10 & -5 \\ -6 & 0 & -5 & -3 \\ -3 & -5 & 0 & -6 \\ -5 & -3 & -6 & 0 \end{bmatrix}. \tag{57}$$

In this example, the type–vector is $(1, 2, 3)$ *and the polygon–vector is* $(0, 3, 6, 3)$, *belonging to class 2. The configuration of the point* $\underline{123}$ *is* $(5.5.5)$ *left and the configuration of* $\underline{4}$ *is* $(6.5.4)$.

We have obtained A *as a perturbation of the anticirculant symmetric matrix* $A(p)$, *with* $p = [0, -6, -3, -5]^t$. *Indeed, in* $\mathrm{span}\, A(p)$, *the configuration is* $(5.6.4)$ *for points* $\underline{123}$ *and* $\underline{4}$. *If the entry* a_{13} *changes from* -3 *to* -10, *then two hyperplanes in figure 13 are crossed, namely those of equations* $a_{12} + a_{43} = a_{42} + a_{13}$ *and* $a_{13} + a_{24} = a_{23} + a_{14}$, *and the class of the tropical tetrahedron changes accordingly.*

Let us finish with a remark. Each configuration at $\underline{123}$ and at $\underline{4}$ are compatible, except for $(5.5.5)$ left and $(5.5.5)$ right. This follows from all the examples above, together with theorems 13, 15 and 14, using symmetry and changes of variables.

Exercises 22.

- *Compute* $\mathrm{tdist}(\underline{ijk}, \underline{ij})$, *for different* i, j, k*; see figure16.*
- *Compare* $\mathrm{span}((A + A')/2)$ *with* $\mathrm{span}(A)$ *and* $\mathrm{span}(A')$, *for different matrices* A, A' *in this paper.*

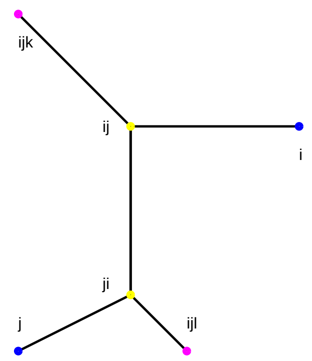

Figure 16: Extremal points in the neighborhood of points $\underline{ij}$ and $\underline{ji}$.

Acknowledgements

We are grateful to our colleague Raquel Díaz for some very useful questions and to S. Sergeev for drawing our attention to Kleene stars.

We have developed several programs in MATLAB for our tropical computations. Factorizations of classical determinants have been done with MAPLE.

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